# NONARCHIMEDEAN FIELDS AND ASYMPTOTIC EXPANSIONS

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A. H. Lightstone &
Abraham Robinson

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# Nonarchimedean Fields and Asymptotic Expansions

A. H. LIGHTSTONE
*Queen's University, Kingston*

and

ABRAHAM ROBINSON †
*Yale University, New Haven*

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*Dedicated to the memory of*
*Abraham Robinson*

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# PREFACE

It has been known for many years that there is a close link between nonarchimedean systems and the orders of infinity and of smallness that are associated with the asymptotic behaviour of a function. The present text provides a background for this connection from the point of view of nonstandard analysis. We have kept the argument at an elementary level and hope that the reader will find the book suitable as an introduction to nonstandard analysis as well as the theory of asymptotic expansions.

The plan of the book is as follows. In the first chapter we introduce the notions of a nonarchimedean group and a nonarchimedean field and give several interesting examples of nonarchimedean fields. Chapter 2 contains an introduction to nonstandard analysis. The necessary resources from mathematical logic are brought in as we go along. In the following two chapters we link up the nonstandard models of analysis, themselves nonarchimedean fields, with a particular nonarchimedean field, here called $\mathcal{L}$, which was first studied by Levi-Civita and Ostrowski and, more recently, by Laugwitz. Unlike the nonstandard models of analysis, $\mathcal{L}$ is canonical (i.e. unique), but unlike the former it cannot be studied by means of a transfer principle. We introduce a natural link between $\mathcal{L}$ and the nonstandard models, the field ${}^{\rho}\mathcal{R}$.

In the last three chapters of the book, we study the fundamentals of asymptotic expansions. Instead of keeping the discussion at a purely theoretical level, we offer a (happy, we hope) mélange of numerical examples and infinitesimals. In sum, we believe that we have at least realized the modest aim of showing that infinitesimals and infinitely large numbers form a natural background to asymptotics.

This monograph is based on a draft by the second author (A.R.), while the final text is due to the first author (A.H.L.). We wish to express our thanks to the North-Holland Publishing Company for agreeing to publish the result of our joint effort in the "Mathematical Library". The first author is indebted to the Canada Council for a Leave Fellowship during 1971–1972, which made this collaboration possible. The second author expresses his thanks to the National Science Foundation for its support over an extended period of time.

A.H. LIGHTSTONE
Queen's University
November 1973

ABRAHAM ROBINSON
Yale University

The manuscript of this book was sent to the publisher in December, 1973, four months before the sudden death of Abraham Robinson. His mastery of mathematics and his warm human qualities will be sorely missed.

A.H. LIGHTSTONE
January 1975

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CHAPTER 1

# NONARCHIMEDEAN FIELDS

## 1. Many-sorted structures

Although the classical version of the Lower Predicate Calculus develops the language of a model, i.e. mathematical system, which possesses just one sort of individual object capable of being quantified, it is clearly a simple matter to extend the Lower Predicate Calculus so that it embraces the more usual mathematical language in which many sorts of individual objects are quantified and related. Moreover, this can be achieved without losing either the strong Completeness Theorem or the Compactness Theorem of mathematical logic.

We shall call mathematical systems of this sort *many-sorted structures*; the language of a many-sorted structure is essentially the usual mathematical language associated with that structure. Accordingly, we shall take full advantage of the abbreviating devices of mathematics in order to communicate more effectively with the reader. On the one hand, we shall benefit from the powerful transfer theorems of mathematical logic by working within a precise, well-defined language which is modelled on a version of the Lower Predicate Calculus known as a *many-sorted calculus*. On the other hand, we shall write down our statements in the language of mathematics, rather than a formal language. However, we can associate with each of our statements a well-formed formula (wff) of the Lower Predicate Calculus.

To illustrate, consider the statement that characterizes a Cauchy sequence $(a_n)$:

$$(1.1) \qquad \forall \epsilon\, \exists q\, \forall mn\, [m,n > q \to |a_m - a_n| < \epsilon].$$

Here Greek letters represent positive real numbers, and Latin letters near the middle of the alphabet represent natural numbers. Using $R$ to denote the real numbers, and $N$ to denote natural numbers (more precisely, the corresponding sets), we find that (1.1) can be expanded as follows:

$$(1.2)\qquad \forall x\, \exists y\, \forall zw\, [x \in R \wedge x > 0 \rightarrow (y \in N \wedge (z \in N \wedge w \in N \wedge z > y \wedge w > y \rightarrow |a_z - a_w| < x))].$$

This leads to a wff of our formal language in a few more steps. To this purpose, we must recognize that the sequence values $a_z$ and $a_w$ can be characterized by a suitable predicate, say $A$. Thus "$Aza$" is interpreted as follows: "the value of the sequence $A$ at $z$ is $a$". So (1.2) expands to

$$(1.3)\qquad \forall x\, \exists y\, \forall zwuv\, [x \in R \wedge x > 0 \rightarrow (y \in N \wedge (z \in N \wedge w \in N \wedge z > y \wedge w > y \wedge Azu \wedge Awv \rightarrow |u - v| < x))].$$

The expression $|u - v|$ may be formalized in terms of a predicate, say $D$; thus "$Duvt$" is interpreted as "the numerical difference of $u$ and $v$ is $t$". Thus (1.3) expands to

$$(1.4)\qquad \forall x\, \exists y\, \forall zwuvt\, [x \in R \wedge x > 0 \rightarrow (y \in N \wedge (z \in N \wedge w \in N \wedge z > y \wedge w > y \wedge Azu \wedge Awv \wedge Duvt \rightarrow t < x))].$$

Finally, we must formalize the unary relations of *being a real number, being a positive real number*, and of *being a natural number*; also, we must formalize the *greater than* relation and the *less than* relation – all of which is routine. The effect of this is to replace "$x \in R$" by "$Rx$", to replace "$y \in N$" by "$Nx$", to replace "$z > y$" by "$>zy$", and to replace "$t < x$" by "$<tx$".

Notice that the above statements refer to a structure involving several sorts of objects, formed into sets which we regard as *supporting* the structure. For example, (1.1) involves the set of all positive real numbers, the set of all natural numbers, and the set of all real numbers itself. We point out that a supporting set of a many-sorted structure can consist of objects that are not numbers, e.g. the set of all sequences, the set of all functions, the set of all subsets of a given set, i.e. its power set. A relation of a many-sorted structure can involve objects from several supporting sets; for example, the relation *the limit of a sequence* can be regarded as a set of ordered pairs whose first terms are sequences and second terms are real numbers – provided that the second term is the limit of the first term.

There is a simple way to transform a many-sorted structure into a *one-sorted* system, i.e., a structure with just one supporting set. Regard each of its supporting sets as a unary relation, and adjoin as its basic set the union of the supporting sets. Thus, let $S_1, ..., S_n$ be the supporting sets of a many-sorted structure $(S_1, ..., S_n, R_1, ..., R_m)$ whose relations are $R_1, ..., R_m$, and let

$$S = S_1 \cup ... \cup S_n;$$

then

$$(S, S_1, ..., S_n, R_1, ..., R_m)$$

is the one-sorted system yielded by $(S_1, ..., S_n, R_1, ..., R_m)$.

For a one-sorted system all quantifiers refer to its basic set, here $S$. Notice that the unary relations $S_1, ..., S_n$ of this one-sorted system allow us, in effect, to quantify over a specific supporting set, rather than over $S$. For example, to quantify over $S_1$ we write

$$\forall x\, [x \in S_1 \rightarrow ...], \qquad \exists x\, [x \in S_1 \wedge ...].$$

The usual mathematical convention, which we shall practise, is to indicate a specific supporting set typographically. Thus, if Greek letters indicate quantification over $S_1$, then we express the above statements by

$$\forall \alpha\, [...], \qquad \exists \alpha\, [...].$$

Another method of streamlining our statements consists in using the language of operations wherever possible. Each operation can be characterized by a relation, and from one viewpoint *is* a relation. For example, *addition* is a binary operation on $R$; so $+$ associates a real number, its sum, with each ordered pair of real numbers. In particular, $+$ associates 5 with (2, 3); this fact is represented by forming the pair ((2, 3), 5), which we can identify with the triple (2, 3, 5). The point is that the set of all triples formed in this way characterizes addition; this set is the relation that represents the operation.

To see the advantage of using the language of operations, rather than relations, consider the associative law for addition. In the language of operations, this law can be stated as

(1.5) $$\forall xyz\, [(x + y) + z = x + (y + z)].$$

This is certainly clear and readable. On the other hand, regarding $+$ as a relation and using the language of relations, our law becomes:

(1.6) $$\forall xyzuvw\, [+xyu \wedge +uzv \wedge +yzw \rightarrow +xwv].$$

Remember, here, that $+abc$ represents the statement $(a, b, c) \in +$.

Notice in (1.5) that we have smuggled in an *equals* sign. The presence of an equality relation is vital to the language of operations. We may characterize equality by a relation of the structure. Of course, an equality relation must, in the first instance, be an equivalence relation (i.e., reflexive, symmetric and transitive). But this is not enough to characterize equality. An equivalence relation requires one additional property in order to serve as equality; it must be *substitutive*. This means that for each relation $T$ of the structure involved and for each positive natural number $n$,

$$(1.7) \qquad \forall x_1 ... x_n y_1 ... y_n \; [Tx_1 ... x_n \wedge = x_1 y_1 \wedge ... \wedge = x_n y_n \rightarrow Ty_1 ... y_n].$$

In short, if $(x_1, ..., x_n) \in T$ and $x_i = y_i$, $i = 1, ..., n$, then $(y_1, ..., y_n) \in T$. This ensures that

$$(x_1, ..., x_n) \in T \quad \text{iff} \quad (y_1, ..., y_n) \in T,$$

provided that $x_i = y_i$ for $i = 1, ..., n$.

We emphasize that equality is itself a relation of the structure involved; so (1.7) must hold with equality in place of $T$ (and $n = 2$); i.e.,

$$(1.8) \qquad \forall x_1 x_2 y_1 y_2 \; [x_1 = x_2 \wedge x_1 = y_1 \wedge x_2 = y_2 \rightarrow y_1 = y_2].$$

Notice that the transitive property of equality follows from (8) and the reflexive property of equality. In short, a binary relation $E$ of a structure is an *equality* relation for that structure provided:

(1) $E$ is substitutive,

(2) $E$ is reflexive,

(3) $E$ is symmetric.

Bear in mind that $E$ is a relation on the *union* of the supporting sets of the structure, not merely on one of its supporting sets.

It follows from (1.7) that any two equality relations of a structure must be the same; i.e., the sets involved have the same members. In this sense, a structure possesses at most one equality relation. On the other hand, a structure without equality can, in general, be provided with several different equality relations; i.e., it can be extended to a structure with equality in several ways. For example, consider

$$(\{0, 1\}, T),$$

where

$$T = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

Here we can take as an equality relation either of the following:

$$E_1 = \{(0,0),(1,1)\}, \qquad E_2 = \{(0,0),(1,1),(0,1),(1,0)\}.$$

Thus $(\{0, 1\}, T, E_1)$ and $(\{0, 1\}, T, E_2)$ are structures with equality. Since $T = E_2$, the latter structure reduces to $(\{0, 1\}, T)$.

If we are given a structure $\mathcal{M}$ in which no relation of equality is specified from the outset, we may always introduce the *identity* as our equality, and we shall do so freely in the sequel. For our purposes, it is best to think of the identity as the *diagonal relation*, i.e., the set of ordered pairs of elements of $M$, the basic set of $\mathcal{M}$, whose first and second terms are the same.

We shall also use the sign of equality in definitions; e.g. to announce that a certain symbol is a name for a specified set. It is easy to distinguish between the equals sign of a definition and the equality relation of a structure; so this practice, which is intended to ease the formal side of this book, should not be confusing.

We shall sometimes refer to a one-sorted system by mentioning its basic set. Generally, however, we shall distinguish between a one-sorted system and its basic set typographically; e.g., if $A$ is the basic set of a one-sorted system, then we shall denote the latter by $\mathcal{A}$.

## 2. Nonarchimedean groups

To prepare the way for nonarchimedean fields, we now present the notion of a *nonarchimedean* group. Recall that a *group* is a structure $(G, +, 0)$, where $+$ is a binary operation (represented by a relation) on $G$, and $0 \in G$, such that:

(1) $\forall xyz\ [x + (y + z) = (x + y) + z]$ ($+$ is associative);

(2) $\forall x\ [x + 0 = x]$ (0 is a right identity);

(3) $\forall x\ \exists y\ [x + y = 0]$ (each group element has a right inverse).

Here $+$ is not necessarily addition and 0 is not necessarily the integer zero.

A group $(G, +, 0)$ is *commutative* or *abelian* if the following statement is true:

(4) $\forall xy\ [x + y = y + x]$ ($+$ is commutative).

We now present some examples.

2.1. EXAMPLE. $(I, +, 0)$ is an abelian group, where $I$ is the set of all integers, $+$ represents addition, and 0 is the integer zero. Here equality means identity; i.e., $a = b$ iff $a$ and $b$ are the same integer. This follows from the fact that if $a$ and $b$ are distinct integers, then so are $a + a$ and $b + a$; thus $(a, a, a + a) \in +$ whereas $(b, a, a + a) \notin +$. So (1.7) is not satisfied.

2.2. EXAMPLE. $(\{0, 1\}, +, 0)$ is an abelian group, where $+$ is

$$\{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}.$$

Here equality is $\{(0, 0), (1, 1)\}$.

2.3. EXAMPLE. $(I, \cdot, 1)$ is *not* a group, where $\cdot$ represents multiplication, since (3) is not true for this structure.

Next we present the notion of an *ordered* set. A structure $(A, <)$, where $<$ is a binary relation on $A$, is an *ordered* set provided that:

(5) $\forall xyz\, [x < y \wedge y < z \to x < z]$ ($<$ is *transitive*);

(6) $\forall xy\, [x < y \to x \neq y]$ ($<$ is *irreflexive*);

(7) $\forall xy\, [x \neq y \to x < y \vee y < x]$ ($<$ is *total*).

As usual, $=$ refers to the identity relation.

We claim that each ordered set $(A, <)$ has the property that

$$\forall xy\, [x < y \to y \nless x].$$

To see this, recall that each statement $p$ in the language of a particular structure is either *true* or *false* for that structure; so exactly one of the statements $p, \neg p$ is true for the structure involved. Thus, if

$$\forall xy\, [x < y \to y \nless x]$$

is not true for an ordered set $(A, <)$, then

$$\neg \forall xy\, [x < y \to y \nless x]$$

*is* true for $(A, <)$; i.e.,

$$\exists xy\, [x < y \wedge y < x]$$

is true for $(A, <)$. This means that there are members of $A$, say $a$ and $b$, such that $a < b$ and $b < a$. But $<$ is transitive, so $a < a$, which contradicts the fact that $<$ is irreflexive.

In view of this, it is easy to establish the *Trichotomy Law* for ordered sets, i.e.,

$$\forall xy\, [\text{exactly one of } x = y, x < y, y < x \text{ is true}].$$

To prove this, let $a, b \in A$. If $a < b$, then $b \nless a$ by the preceding paragraph, and $a \neq b$ by (6). If $a = b$, then by (6), $a \nless b$. Again, if $a = b$, then $b = a$, so $b \nless a$. Thus at most one of the three statements

$$a = b, \qquad a < b, \qquad b < a$$

is true. Moreover, (7) guarantees that at least one of these three statements is true. This establishes the Trichotomy Law.

Here are some examples.

2.4. EXAMPLE. $(N, <)$ is an ordered set, where $N$ is the set of all natural numbers and $<$ is the less than relation on $N$.

2.5. EXAMPLE. $(\{0, 1\}, <)$ is an ordered set, where $<$ is $\{(0, 1)\}$.

2.6. EXAMPLE. $(N, \leqslant)$ is *not* an ordered set; notice that (6) is false for this structure.

Building on these ideas, it is easy to characterize the notion of an *ordered* group; this is a structure of the form $(G, +, <, 0)$ such that $(G, +, 0)$ is an abelian group, $(G, <)$ is an ordered set, and the order relation $<$ is compatible with the group operation $+$; i.e.,

(8) $\forall xyz\ [x < y \to x + z < y + z]$.

Here are some examples.

2.7. EXAMPLE. $(I, +, <, 0)$ is an ordered group.

2.8. EXAMPLE. $(I, +, >, 0)$ is an ordered group.

2.9. EXAMPLE. $(\{0, 1\}, +, <, 0)$ is *not* an ordered group, where $+$ is the operation of Example 2.2 and $<$ is the order relation of Example 2.5.

Notice that we use the symbol $+$ as a generic or family name for a group operation; similarly, we use the symbol $<$ as a generic or family name for an order relation. In the same spirit, we use the symbol 0 to denote the identity element of a group (in case the group operation is denoted by $+$).

Now an ordered group $(G, +, <, 0)$ is said to have the *archimedean* property if

(9) $\forall xy\ [0 < x < y \to y < x + \ldots + x\ (n\ x\text{'s})]$,

where $n$ is a positive natural number that depends on the members of $G$ involved. Let us represent the expression "$x + \ldots + x$" by writing $nx$ if there are $n$ $x$'s in the expression. Then (9) simplifies to

(10) $\forall xy\ \exists n\ [0 < x < y \to y < nx]$.

Here the existential quantifier refers to $N$, the set of all natural numbers; of course, the universal quantifiers refer to $G$, the basic set of the ordered group.

Notice that the ordered groups of Examples 2.7 and 2.8 possess the archimedean property. An ordered group that satisfies (9) is called an *archimedean* group; on the other hand, an ordered group for which (9) is false is called a *nonarchimedean* group. We now present an example of a nonarchimedean group.

2.10. EXAMPLE. We shall construct a nonarchimedean group $(G, +, <, 0)$. Take $G = I \times I$, the set of all ordered pairs whose terms are integers, and let $+$ be the operation on $G$ of adding corresponding terms, i.e.

$$(a, b) + (c, d) = (a + c, b + d).$$

Here the $+$ on the left is the operation on $G$ that we are defining and the $+$'s on the right represent addition of integers. Next define an order relation $<$ on $G$ as follows: $(a, b) < (c, d)$ if either

(a) $a < c$ or

(b) $a = c \wedge b < d$.

Here $<$ is the less than relation on the integers. Take $0 = (0, 0)$. Notice that the inverse of $(a, b)$ is $(-a, -b)$. It is easy to verify that $(G, +, <, 0)$ is an ordered group. To see that this ordered group does not have the archimedean property, notice that

$$(0, 0) < (0, 1) < (1, 0), \qquad (0, n) < (1, 0) \quad \text{for each } n \in N.$$

We point out that $n(0, 1) = (0, n)$. Thus our ordered group does not have the archimedean property; so $(G, +, <, 0)$ is a nonarchimedean group.

## 3. Nonarchimedean fields

A *field* is a structure of the form $(F, +, \cdot, 0, 1)$, where $(F, +, 0)$ and $(F - \{0\}, \cdot, 1)$ are abelian groups, and the operations $+$ and $\cdot$ have the following properties:

(1) $\forall x\, [x \cdot 0 = 0 \cdot x = 0]$;

(2) $\forall xyz\, [x \cdot (y + z) = x \cdot y + x \cdot z]$ (a *distributive* law).

Here the quantifiers refer to $F$.

Since $1 \in F - \{0\}$, it follows that $1 \neq 0$. We point out that $\cdot$ is a binary operation on $F$; in the case of the abelian group $(F - \{0\}, \cdot, 1)$ we are dealing with its *restriction* to $F - \{0\}$. By assumption, $\cdot$ is associative on $F - \{0\}$; it follows from (1) that $\cdot$ is associative on $F$. Similarly we see that $\cdot$ is commutative on $F$.

We present some examples.

3.1. EXAMPLE. Let $Q$ be the set of all rational numbers, interpret $+$ and $\cdot$ as addition and multiplication of rational numbers, and interpret 0 and 1 as the usual rational numbers. Then $(Q, +, \cdot, 0, 1)$ is a field.

3.2. EXAMPLE. Let $F = Q \times Q$, the set of all ordered pairs whose terms are rational numbers. Define operations $+$ and $\cdot$ on $F$ by

$$(a, b) + (c, d) = (a + c, b + d),$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc),$$

where the operations on the RHS refer to the field of Example 3.1. Let $0 = (0, 0)$, and let $1 = (1, 0)$. It is easy to verify that the structure $(F, +, \cdot, 0, 1)$ is a field. Notice that we have aped the construction of the complex number field from the real number field. Indeed, the field of this example is a subfield of the complex number field.

Next we introduce an *order* relation into the picture. A structure of the form $(F, +, \cdot, <, 0, 1)$ is said to be an *ordered field* if:

(3) $(F, +, \cdot, 0, 1)$ is a field;

(4) $(F, +, <, 0)$ is an ordered group;

(5) $\forall xyz\ [x < y \wedge 0 < z \rightarrow x \cdot z < y \cdot z]$ ($<$ is *compatible* with $\cdot$).

For example, $(Q, +, \cdot, <, 0, 1)$ is an ordered field, where $<$ is the usual less than relation on $Q$; also, $(R, +, \cdot, <, 0, 1)$ is an ordered field, where $R$ is the set of all real numbers, $+$ and $\cdot$ are the usual operations of addition and multiplication, $<$ is the less than relation on $R$, and 0 and 1 denote the corresponding real numbers

3.3. EXAMPLE. Here is an example of a structure that satisfies (3) and (4), but not (5). We adjoin to the field $(Q \times Q, +, \cdot, 0, 1)$ of Example 3.2 the binary relation $<$ defined as follows: $(a, b) < (c, d)$ if either

(a) $a < c$ or

(b) $a = c \wedge b < d$,

where the $<$ appearing in (a) and (b) is the less than relation on $Q$. Clearly, $(Q \times Q, <)$ is an ordered set; indeed, $(Q \times Q, +, <, 0)$ is an ordered group. So conditions (3) and (4) are met. To see that $<$ is *not* compatible with $\cdot$, notice that $(0, 0) < (0, 1)$. We shall multiply by $(0, 1)$; now $(0, 0) \cdot (0, 1) = (0, 0)$ and $(0, 1) \cdot (0, 1) = (-1, 0)$, but $(0, 0) < (-1, 0)$ is false.

Now an ordered field $(F, +, \cdot, <, 0, 1)$ is said to have the *archimedean* property if the ordered group $(F, +, <, 0)$ has the archimedean property; i.e.,

(6) $\forall xy\, \exists n\, [0<x<y \to y<nx]$,

where the universal quantifier refers to $F$, the existential quantifier refers to $N$, and $nx = x + \ldots + x$ ($n$ $x$'s).

We shall call an ordered field that possesses the archimedean property, an *archimedean field*; and we shall call an ordered field that does not have this property a *nonarchimedean field*. In other words, a nonarchimedean field is an ordered field such that

(7) $\exists xy\, \forall n\, [0<x<y \wedge nx \leqslant y]$,

where the existential quantifier refers to the basic set of the ordered field, the universal quantifier refers to $N$, and we have followed the usual custom of abbreviating $p<q \vee p=q$ by $p \leqslant q$.

The ordered fields $(Q, +, \cdot, <, 0, 1)$ and $(R, +, \cdot, <, 0, 1)$ mentioned above are both archimedean fields. Before illustrating the notion of a nonarchimedean field, we present a simple method of characterizing the order relation of an ordered field. Let $(F, +, \cdot, <, 0, 1)$ be any ordered field; let $P' = \{x \mid 0<x\}$, the set of all positive field elements. This set has the following properties:

(8) $0 \notin P'$;

(9) $\forall x\, [x \neq 0 \to x \in P' \vee -x \in P']$;

(10) $\forall xy\, [x,y \in P' \to x+y \in P' \wedge x \cdot y \in P']$.

The point is that each subset of $F$, say $P$, that has these three properties yields an order relation on $F$ which is compatible with both $+$ and $\cdot$; this order relation, which we denote by $<$, is defined as follows:

$$x<y \quad \text{if and only if} \quad y-x \in P.$$

In other words, we can get at an order relation by first defining $P$, the set of its positive elements.

Notice that if $P$ is a subset of $F$ satisfying (8), (9) and (10), then $P$ also satisfies the following statement:

(11) $\forall x\, [x \in P \to -x \notin P]$.

In view of (11) we can replace (9) by

(9′) $\forall x\, [x \neq 0 \to x \in P \veebar -x \in P]$,

where $\veebar$ is the "exclusive or".

Here are some elementary properties of ordered fields. First we mention that $0<1$; otherwise, by (9′), $0<-1$, and by the Trichotomy Law, $1<0$. Thus, by (5), $1 \cdot (-1) < 0 \cdot (-1)$, i.e., $-1<0$, which contradicts the Trichotomy Law.

Since $1 \in P$, it follows from (10) that $1+1 \in P$, indeed that $n1 \in P$ for each $n \in N$, $n>0$. Of course, this means in particular that $n1 \neq 0$ for each $n \in N$, $n>0$; so each ordered field has characteristic zero.

Notice that $x^2 \in P$ if $x \neq 0$. If $x \in P$, apply (10). If $x \notin P$, then $-x \in P$ by (9′), so $(-x)^2 \in P$ by (10). But

$$(-x)^2 = (-1)^2 \cdot x^2 = x^2$$

since $(-1)^2 = 1$. Remember that $(-1) \cdot a = -a$, the additive inverse of $a$, so

$$(-1)^2 = (-1) \cdot (-1) = -(-1) = 1$$

(since 1 is the additive inverse of $-1$).

We now present an example of a nonarchimedean field.

3.4. EXAMPLE. Let $f$ be a polynomial function over $R$; i.e., $f$ is a map of $R$ into $R$ such that

$$f(t) = a_0 + a_1 t + \ldots + a_n t^n$$

for each $t \in R$, where $a_0, \ldots, a_n \in R$ and $n \in N$. Thus each polynomial function has associated with it a natural number $n$, called its *degree*, and $n + 1$ fixed real numbers $a_0, \ldots, a_n$ (not necessarily distinct). More simply, we can characterize a polynomial function of degree $n$ by the $(n+1)$-tuple $(a_0, \ldots, a_n)$. Let $g$ be a nonzero polynomial function, i.e., $g(t) \neq 0$ for some $t \in R$. Then the formal expression $f/g$ represents the rational function that associates the quotient $f(t)/g(t)$ [i.e. $f(t) \div g(t)$] with $t$, where $t \in R$ and $g(t) \neq 0$. Let $F$ be the set of all rational functions $f/g$ such that $f$ and $g$ are relatively prime (any common factors have been cancelled out) and such that the trailing coefficient of $g$ (the coefficient of the lowest power of $t$ in $g$) is 1; i.e.,

$$g(t) = t^j + b_{j+1} t^{j+1} + \ldots + b_m t^m \quad \text{for each } t \in R,$$

where $b_{j+1}, \ldots, b_m \in R$ and $j \in N$. So $g = (b_0, \ldots, b_m)$, where 1 is the first nonzero term of $g$. Define addition and multiplication on $F$ as for fractions; i.e.,

$$f/g + p/q = (fq + gp)/gq, \qquad f/g \cdot p/q = fp/gq,$$

where each RHS is to be reduced to a member of $F$ by cancelling out common factors and multiplying through by the multiplicative inverse of the trailing coefficient of the denominator. Notice in these definitions that the operations of addition and multiplication that occur in the RHS are operations on polynomial functions; so addition and multiplication on $F$ are defined in terms of the corresponding operations on polynomial functions.

Clearly, $0 = \{(t, 0) \mid t \in R\}$ is the additive identity, and $1 = \{(t, 1) \mid t \in R\}$ is the multiplicative identity. Thus we obtain the field $(F, +, \cdot, 0, 1)$.

We shall get at an order relation on $F$ by defining its positive elements. Let $f/g \in P$ (i.e., $f/g$ is positive) if the trailing coefficient of $f$ is positive. Notice that $P$ satisfies condition (8), (9) and (10); so $P$ yields an order relation $<$ on $F$. We claim that the resulting ordered field $(F, +, \cdot, <, 0, 1)$ is nonarchimedean. To see this, let

$$x = \{(t, t) \mid t \in R\},$$

the *identity* function; then $x \in P$ and $1 - nx \in P$ for each $n \in N$. Thus $0 < x < 1$ and $nx < 1$ for each $n \in N$.

Let $\mathcal{F} = (F, +, \cdot, <, 0, 1)$ be any ordered field. Here we define *absolute value*; as for the reals, this can be expressed in terms of $P$, the set of all positive elements of $F$, as follows:

$$|a| = \begin{cases} a & \text{if } a \in P, \\ -a & \text{if } a \notin P. \end{cases}$$

In particular, $|0| = -0 = 0$. As usual, the *absolute value* has the following properties:

(12) $\forall x\, [|x| \geqslant 0]$;

(13) $\forall xy\, [|xy| = |x|\,|y|]$;

(14) $\forall xy\, [|x + y| \leqslant |x| + |y|]$ (Triangle Inequality).

Suppose that $\mathcal{F}$ is a nonarchimedean field. Then $\mathcal{F}$ has characteristic zero, so we can regard this field as an extension of the rational number field. Let $Q$ be the subset of $F$ whose members are identified with the rational numbers. We propose to characterize the infinitely large elements of $F$ and its infinitely small elements. We say that $a \in F$ is *infinite* (or *infinitely large*) if $|a| > q$ for each $q \in Q$; we say that $a \in F$ is an *infinitesimal* (or *infinitely small*) if $|a| < q$ for each positive $q \in Q$. We say that $a \in F$ is *finite* if $a$ is not infinite.

Here is a useful way of characterizing the infinitesimals of a nonarchimedean field $\mathcal{F}$. First observe that $n1 \in F$ for each $n \in N$. Clearly $a$ is an infinitesemial iff $\forall n\, [n\,|a| < 1]$. Similarly we can characterize the infinitely large members of $F$: $a$ is infinite iff $\forall n\, [n1 < |a|]$.

Now the sum and product of two finite elements of $F$ are also finite; indeed

$$F_0 = \{a \mid a \in F \wedge a \text{ is finite}\}$$

forms a subring of $\mathcal{F}$. Also, the sum of two infinitesimals is an infinitesimal, and the product of a finite element and an infinitesimal is an infinitesimal; thus

$$F_1 = \{a \mid a \in F \wedge a \text{ is an infinitesimal}\}$$

is a proper ideal of the ring $F_0$.

It is important to observe that $F_1$ is a maximal ideal of $F_0$. To see this, notice that if $a$ is not an infinitesimal, then its multiplicative inverse $a^{-1}$ is not infinite, i.e., $a^{-1} \in F_0$. So, if there is an ideal of $F_0$, say $J$, such that $F_1$ is a proper subset of $J$, then there is a member of $J$, say $a$, such that $a \notin F_1$; so $a^{-1} \in F_0$. Therefore $a \cdot a^{-1} \in J$, i.e. $1 \in J$, and it follows that $J = F_0$. This proves that $F_1$ is maximal in $F_0$. By Lemma 3.1.2, $F' = F_0/F_1$ is a field, called the *residue class field of the ordering*. Here the field elements are the cosets of $F_1$ in $F_0$, namely

$$[b] = \{b + x \mid x \in F_1\},$$

where $b \in F_0$.

More generally, the cosets of $F_1$ in $F$ are called *monads*. Let $a \in F$. Then

$$\mu(a) = [a] = \{a + x \mid x \in F_1\}$$

is called the *monad of a*. In particular, $\mu(0) = F_1$. Notice that $F_1$ is the additive identity of the field $F_0/F_1$.

Let $[t] \neq F_1$; we want to prove that each member of $[t]$ is positive (in $\mathcal{F}$), or each member of $[t]$ is negative (in $\mathcal{F}$). To see this, let $P$ be the set of all positive members of $F$, and let $s \in [t]$. If $t \in P$ and $-s \in P$, then $0 < -s$, so $t + 0 < t - s$, i.e. $t - s > t$. But $t \geqslant q$ for some $q \in Q$; thus $t - s \notin F_1$. This contradiction proves that all members of $[t]$ have the same algebraic sign.

We are now in a position to introduce an order relation on the set $F_0/F_1$. Let $[t] \neq F_1$. We say that $[t]$ is *positive* (in $\mathcal{F}'$) iff $t$ is positive (in $\mathcal{F}$). By our preceding observation, $t$ is positive iff each member of $[t]$ is positive. Let $P'$ be the set of all positive members of $F_0/F_1$. Then $P'$ has the following properties:

(15) $0 \notin P'$;

(16) $\forall x\, [x \neq 0 \rightarrow x \in P' \vee -x \in P']$;

(17) $\forall xy\, [x,y \in P' \rightarrow x + y \in P' \wedge xy \in P']$.

Accordingly, the associated order relation $<$, for which $x < y$ iff $y - x \in P'$, is compatible with both addition and multiplication of the residue class field $\mathcal{F}'$. In this way, $\mathcal{F}'$ extends to an ordered field.

Moreover, this ordered field is archimedean. To see this, let $[t]$ and $[s]$ be positive. We must prove that there is a natural number $n$ such that $n[t] > [s]$. By assumption, both $s$ and $t$ are finite, and neither $s$ nor $t$ is an infinitesimal of $\mathcal{F}$. Therefore there are positive rationals $q_0$ and $q_1$ such that $s \leqslant q_0$ and $q_1 < t$. But there is a natural number $n$ such that $nq_1 > q_0$; thus

$$nt > nq_1 > q_0 \geqslant s,$$

so $nt > s$. We conclude that the residue class field $F_0/F_1$ is archimedean with respect to the order relation $<$.

It is well known that each archimedean field is isomorphic to a subfield of the real number field $\mathcal{R}$. So, in this sense, the finite elements of a nonarchimedean field yield a subfield of $\mathcal{R}$ by factoring out its infinitesimals.

## 4. Fields with valuation

A useful tool in the investigation of nonarchimedean fields is the notion of a *nonarchimedean valuation* on a field $\mathcal{F} = (F, +, \cdot, 0, 1)$. A nonarchimedean valuation on $\mathcal{F}$ is a map $v$ of $F$ into $R \cup \{\infty\}$, where $\infty$ is an additional element called *infinity*, such that:

(1) $\forall x\ [x \neq 0 \rightarrow v(x) \in R],\ v(0) = \infty$;

(2) $\forall xy\ [v(x \cdot y) = v(x) + v(y)]$;

(3) $\forall xy\ [v(x+y) \geqslant \min\{v(x), v(y)\}]$.

Here quantification is over $F$. We assume that the number system involving $R \cup \{\infty\}$ has the properties

$$\forall x\ [x + \infty = \infty], \qquad \forall x\ [x \leqslant \infty],$$

where quantification is over $R \cup \{\infty\}$.

From (2),

$$v(1) = v(1) + v(1),$$

so $v(1) = 0$. So, from (2),

$$0 = v(x \cdot x^{-1}) = v(x) + v(x^{-1}) \quad \text{if } x \neq 0.$$

Therefore,

$$\forall x\ [x \neq 0 \rightarrow v(x^{-1}) = -v(x)].$$

Since $(-1) \cdot (-1) = 1$ in $\mathcal{F}$, it follows from (2) that

$$0 = v(1) = v(-1 \cdot -1) = v(-1) + v(-1),$$

so $v(-1) = 0$. Now, for each $x \in F$,

$$-x = (-1) \cdot x;$$

so, by (2),

$$v(-x) = v(-1 \cdot x) = v(-1) + v(x) = v(x),$$

i.e.,

$$\forall x\ [v(-x) = v(x)].$$

By the *trivial valuation* on $\mathcal{F}$ we mean the map that associates 0 with each nonzero member of $F$, and associates $\infty$ with 0. Clearly this map is a nonarchimedean valuation on $\mathcal{F}$.

Here is another example.

4.1. EXAMPLE. Each nonzero rational number $x$ can be written in the form

$$x = (a/b)\, p^{\alpha},$$

where $p$ is a fixed prime, $\alpha \in I$, and $a$ and $b$ are integers that are not divisible by $p$. Let $v_p$ be the mapping of $Q$ into $R \cup \{\infty\}$ for which $v_p(0) = \infty$ and $v_p(x) = \alpha$ if $x = (a/b)\,p^{\alpha}$ as above. It is easy to verify that the map $v_p$ satisfies (1), (2) and (3); so $v_p$ is a nonarchimedean valuation on the rational number field $Q$. The valuation of this example is called a *p-adic* valuation.

In our next example, we introduce a field of Laurent series. Throughout this book, $N = \{0, 1, 2, ...\}$; we sometimes write $\Sigma_N$ in place of $\Sigma_{n \in N}$.

4.2. EXAMPLE. Let $(F, +, \cdot, <, 0, 1)$ be any ordered field. A *Laurent* series is a formal object

$$\sum_{n \in N} a_{n+k}\, t^{n+k},$$

where $k$ is a fixed integer (i.e., fixed for this Laurent series), each $a \in F$, and either $a_k \neq 0$ or each $a = 0$. We shall identify any two Laurent series of the latter sort (i.e., each coefficient is zero) and shall denote each of these Laurent series by 0. Also, we shall identify each expression of the form

$$\sum_{n \in N} a_{n+i}\, t^{n+i},$$

where $0 = a_i = ... = a_{k-1}$ and $a_k \neq 0$ for some integer $k$ (these expressions are not Laurent series), with the Laurent series $\Sigma_N\, a_{n+k}\, t^{n+k}$. The Laurent series $\Sigma_N\, a_n\, t^n$, where $a_0 = 1$ and $a_n = 0$ if $n > 0$, is denoted by 1.

Addition of Laurent series is defined by adding corresponding coefficients, after lining up powers of $t$. Thus

$$\sum_N a_{n+k}\, t^{n+k} + \sum_N b_{n+j}\, t^{n+j} = \sum_N c_{n+i}\, t^{n+i},$$

where $i = \min\{k, j\}$, and for $m \geqslant i$, $c_m = a'_m + b'_m$, where $a'_m = a_m$ if $m \geqslant k$, otherwise $a'_m = 0$, and $b'_m = b_m$ if $m \geqslant j$, otherwise $b'_m = 0$.

Multiplication is defined as follows:

$$\Big(\sum_N a_{n+k}\,t^{n+k}\Big)\Big(\sum_N b_{n+j}\,t^{n+j}\Big) = \sum_N c_{n+i}\,t^{n+i},$$

where $i = kj$, and

$$\begin{aligned} c_i &= a_k b_j, \\ c_{i+1} &= a_k b_{j+1} + a_{k+1} b_j, \\ c_{i+2} &= a_k b_{j+2} + a_{k+1} b_{j+1} + a_{k+2} b_j, \end{aligned}$$

etc. It is well known that the resulting structure is a field.

We introduce an order relation on Laurent series over $\mathcal{F}$ by defining a nonzero Laurent series $\Sigma_N\, a_{n+k} t^{n+k}$ to be *positive* iff its first coefficient is positive in the underlying ordered field; i.e. $\Sigma_N\, a_{n+k}\, t^{n+k} \in P$ iff $a_k > 0$. Clearly $P$ satisfies conditions (8), (9) and (10) of Section 3; so the Laurent series over $\mathcal{F}$ constitute an ordered field.

Finally, we introduce a nonarchimedean valuation $v$ on the field of Laurent series as follows. Define

$$v(0) = \infty$$

and

$$v\Big(\sum_N a_{n+k}\,t^{n+k}\Big) = k \quad \text{if } \sum_N a_{n+k}\,t^{n+k} \neq 0$$

(so $a_k \neq 0$). It is easy to verify that $v$ is a nonarchimedean valuation on this field.

Returning to our general theory, we now present a basic fact about nonarchimedean valuations.

4.3. LEMMA. $\forall xy\, [v(x) < v(y) \rightarrow v(x+y) = v(x)]$.

*Proof.* By (3), $v(x+y) \geqslant v(x)$. We shall show that $v(x+y) \leqslant v(x)$. Now

$$x = (x+y) - y,$$

so

$$v(x) \geqslant \min\{v(x+y), v(-y)\} = \min\{v(x+y), v(y)\}$$

since $v(-y) = v(y)$. If $v(y) \leqslant v(x+y)$, then $v(x) \geqslant v(y)$. This contradicts our assumption that $v(x) < v(y)$; so $v(x+y) < v(y)$, thus $v(x) \geqslant v(x+y)$. This proves that $v(x+y) = v(x)$. □

We mention that Lemma 4.3 is usually stated in the following form:

$$\forall xy\ [v(x) \neq v(y) \rightarrow v(x+y) = \min\{v(x), v(y)\}].$$

We shall now define a few terms. First notice that

$$\{v(x) \mid x \in F \wedge x \neq 0\}$$

is an additive subgroup of $\mathcal{F}$; this set is called the *valuation group* of $v$. The set

$$O_F = \{x \mid x \in F \wedge v(x) \geqslant 0\}$$

is a subring of $\mathcal{F}$ and is called the *valuation ring* of $v$. The set

$$J_F = \{x \mid x \in F \wedge v(x) > 0\}$$

is a maximal ideal of the ring $O_F$, and is called the *valuation ideal* of $v$. The set

$$U_F = \{x \mid x \in F \wedge v(x) = 0\}$$

is a multiplicative subgroup of $O_F$, and is called the *group of units* of $v$. The field $\bar{F} = O_F/J_F$ is called the *residue class field* of $v$.

A map $v$ of $F$ into $R \cup \{\infty\}$ that satisfies (1) and (2) and the statement

(3′) $\forall xy\ [\mathrm{e}^{-v(x+y)} \leqslant \mathrm{e}^{-v(x)} + \mathrm{e}^{-v(y)}]$,

and does not satisfy (3), is said to be an *archimedean* valuation. If you are curious about the presence of the exponential function in (3′), look ahead to Section 5.

Here is an example of an archimedean valuation.

4.4. EXAMPLE. We present an archimedean valuation on the real number field $\mathcal{R}$. Let $v$ be the map of $R$ into $R \cup \{\infty\}$ such that

$$v(0) = \infty, \qquad v(a) = -\ln|a| \quad \text{if } a \neq 0.$$

Notice that for each $x \in R$, $\mathrm{e}^{-v(x)} = |x|$; here $\mathrm{e}^{-\infty}$ is interpreted as 0. Clearly (1), (2) and (3′) are satisfied. To see that (3) is false for $v$, observe that

$$v(\mathrm{e}+\mathrm{e}) = -\ln|2\mathrm{e}| = -\ln 2 - 1 < -1,$$

whereas $v(\mathrm{e}) = -\ln \mathrm{e} = -1$; so $v(\mathrm{e}+\mathrm{e}) < \min\{v(\mathrm{e}), v(\mathrm{e})\}$.

## 5. Development of metric

A valuation on a field $\mathcal{F}$ may be used to build up a metric on $\mathcal{F}$. This is achieved in two steps. First we define a mapping $|\ |_v$, in terms of the given valuation $v$, as follows:

$$|x|_v = e^{-v(x)} \quad \text{for each } x \in F$$

(in place of e we can use any number greater than 1), where "$e^{-\infty}$" is interpreted as the real number zero. In particular, $|\ |_v$ is a map of $F$ into $R$. In case $v$ is a *nonarchimedean* valuation, this map has the following properties:

(1) $\forall x\ [x \neq 0 \rightarrow |x|_v > 0], |0|_v = 0, |1|_v = 1;$
(2) $\forall x\ [|-x|_v = |x|_v];$
(3) $\forall xy\ [|x-y|_v = |y-x|_v];$
(4) $\forall xy\ [|xy|_v = |x|_v |y|_v];$
(5) $\forall xy\ [|x+y|_v \leqslant \max\{|x|_v, |y|_v\} \leqslant |x|_v + |y|_v];$
(6) $\forall xy\ [|y|_v < |x|_v \rightarrow |x+y|_v = |x|_v];$
(7) $\forall xyz\ [|x-y|_v > |y-z|_v \rightarrow |x-z|_v = |x-y|_v].$

These statements can be established directly from the properties of $v$ listed in Section 4 and the fact that the function exp is monotonically increasing. For example, to prove (6) observe that

$$|y|_v < |x|_v \quad \text{iff} \quad e^{-v(y)} < e^{-v(x)} \quad \text{iff} \quad e^{v(x)} < e^{v(y)} \quad \text{iff} \quad v(x) < v(y).$$

So by Lemma 4.3,

$$v(x+y) = v(x);$$

thus $e^{-v(x+y)} = e^{-v(x)}$, i.e., $|x+y|_v = |x|_v$.

Certainly, (7) is a striking statement. This fact can be deduced from (6) as follows:

$$|x-z|_v = |(x-y)+(y-z)|_v = \max\{|x-y|_v, |y-z|_v\} = |x-y|_v.$$

It follows from (4) that

$$\forall x\ [x \neq 0 \rightarrow |1/x|_v = 1/|x|_v].$$

We are now ready to introduce a metric $d$ on $\mathcal{F}$, defined in terms of the mapping $|\ |_v$. Let $d$ be the map of $F \times F$ into $R$ such that for all $x,y \in F$,

$$d(x,y) = |x-y|_v.$$

We must show that $d$ has the following properties:

(8) $\forall x\ [d(x,x) = 0];$
(9) $\forall xy\ [x \neq y \rightarrow d(x,y) > 0];$
(10) $\forall xy\ [d(x,y) = d(y,x)];$
(11) $\forall xyz\ [d(x,z) \leqslant d(x,y) + d(y,z)].$

Now, for each $x \in F$,

$$d(x,x) = |x - x|_v = |0|_v = 0$$

by (1), and if $x \neq y$, then

$$d(x,y) = |x - y|_v > 0$$

by (1). Notice that (10) follows from (3). To prove (11), the Triangle Inequality, observe that

$$d(x,z) = |x - z|_v = |(x - y) + (y - z)|_v \leqslant |x - y|_v + |y - z|_v$$

$$= d(x,y) + d(y,z)$$

by (5). So $(F, d)$ is a metric space.

From (7), if $d(x,y) > d(y,z)$, then $d(x,z) = d(x,y)$; so each triangle in the metric space $(F, d)$ is "isosceles".

Moreover,

$$\forall xy \; [|x + y|_v \leqslant \max\{|x|_v, |y|_v\}];$$

so

$$d(x,z) = |x - z|_v = |(x - y) + (y - z)|_v \leqslant \max\{|x - y|_v, |y - z|_v\}$$

$$= \max\{d(x,y), d(y,z)\}.$$

This establishes the following statement, which is known as the *ultrametric inequality*:

(12) $\forall xyz \; [d(x,z) \leqslant \max\{d(x,y), d(y,z)\}]$.

So the metric constructed from a nonarchimedean valuation satisfies the ultrametric inequality.

Next we shall use a metric to define the notion of a *convergent* sequence in $\mathcal{F}$ and the notion of the *limit* of a convergent sequence. This is carried out by appealing to the corresponding concepts for the real number field $\mathcal{R}$. Recall that a sequence of real numbers $(a_n)$ converges in $\mathcal{R}$ if there is a real number $a$ such that

$$\forall \epsilon \, \exists n \, \forall m \; [m > n \rightarrow |a_m - a| < \epsilon],$$

where the Greek letter represents a positive real number, $m, n \in N$, and the absolute value signs denote the usual absolute value in $\mathcal{R}$. Moreover, $a$ is called the limit of $(a_n)$, and we write "$\lim(a_n) = a$". Of course, a sequence of real numbers is a map of $N$ into $R$.

We now formulate corresponding ideas in the field $\mathcal{F}$. First, by a *sequence* of field elements we mean any map of $N$ into $F$, say $(s_n)$, where $s_n \in F$ for each $n \in N$. We define *convergence* in terms of a metric $d$ as follows: $(s_n)$ converges to $s$, where $s \in F$, provided that the real sequence $(d(s_n, s))$ converges to 0, i.e. $\lim(d(s_n, s)) = 0$. In this case we say that $s$ is the *limit* of $(s_n)$ and write "$\lim(s_n) = s$". Each convergent sequence has a unique limit.

Here is a useful fact.

5.1. THEOREM. *If* $\lim(s_n) = s$, *then* $\lim(|s_n|_v) = |s|_v$. *Here* $v$ *is nonarchimedean.*

*Proof.* Let $(s_n)$ be a sequence that converges to $s$. This means that $\lim(d(s_n, s)) = 0$; i.e., $\lim(|s_n - s|_v) = 0$. There are two cases:

(i) Assume $s = 0$. Then $\lim(|s_n|_v) = 0$. But $|0|_v = e^{-\infty} = 0$, by definition; thus $\lim(|s_n|_v) = |s|_v$.

(ii) Assume $s \neq 0$. Then $|s|_v$ is positive and real. But $\lim(|s_n - s|_v) = 0$; so there is a natural number $q$ such that for each $m > q$, $m \in N$,

(5.2) $$|s_m - s|_v < |s|_v.$$

By (6) it follows from (5.2) that for each $m > q$,

(5.3) $$|s + (s_m - s)|_v = |s|_v,$$

i.e. $|s_m|_v = |s|_v$. We conclude that $\lim(|s_n|_v) = |s|_v$. □

A sequence $(s_n)$ is called a *Cauchy* sequence if

$$\forall \epsilon \, \exists n_0 \, \forall mn \, [m,n > n_0 \rightarrow d(s_m, s_n) < \epsilon].$$

Each convergent sequence is a Cauchy sequence; however, it is not necessarily the case that each Cauchy sequence converges. If each Cauchy sequence in $\mathcal{F}$ converges, we say that the field $\mathcal{F}$ is *complete* with respect to the metric $d$; in case the metric $d$ is yielded by a valuation $v$, we say that $\mathcal{F}$ is complete with respect to $v$.

Here are some examples.

5.4. EXAMPLE. The trivial valuation on a field $\mathcal{F}$ yields the metric $d$ for which

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Here a sequence $(s_n)$ is a Cauchy sequence iff

$$\exists i \, \forall m \, [m > i \rightarrow s_m = s_i];$$

i.e., all the terms of $(s_n)$ beyond a certain term are the same, say $s$. Clearly, in this case, $\lim(s_n) = s$; thus $\mathcal{F}$ is complete with respect to the trivial valuation.

5.5. EXAMPLE. The rational number field $Q$ is *not* complete with respect to the usual metric $d$, i.e.,

$$d(x,y) = |x - y| \quad \text{for all } x,y \in Q.$$

For example, it is well known that the sequence (1.4, 1.41, 1.414, ...), whose $n^{\text{th}}$ term is obtained by truncating $\sqrt{2}$ to $n$ decimal places, is a Cauchy sequence but does not converge in $Q$.

Notice that if $v$ is a nonarchimedean valuation on $\mathcal{F}$, then for any $x_1,\dots,x_n \in F$,

$$|x_1 + \dots + x_n|_v \leqslant \max\{|x_1|_v, \dots, |x_n|_v\}.$$

Using this fact, it is easy to prove our next lemma.

5.6. LEMMA. *Let $v$ be a nonarchimedean valuation on $\mathcal{F}$. Then $(a_n)$ is a Cauchy sequence with respect to $v$ iff* $\lim(a_{n+1} - a_n) = 0$.

*Proof.* Notice that $\lim(a_{n+1} - a_n) = 0$, the zero of $\mathcal{F}$, iff $\lim(|a_{n+1} - a_n|_v) = 0$, the zero of $\mathcal{R}$. If $(a_n)$ is a Cauchy sequence, then in particular $|a_{n+1} - a_n|_v < \epsilon$ if $n > n_0$; so $\lim(|a_{n+1} - a_n|_v) = 0$.

Next assume that $\lim(|a_{n+1} - a_n|_v) = 0$. Now

$$a_m - a_n = (a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \dots + (a_{n+1} - a_n),$$

where we assume that $m > n$; so

$$\begin{aligned} |a_m - a_n|_v &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \dots + (a_{n+1} - a_n)|_v \\ &\leqslant \max\{|a_m - a_{m-1}|_v, |a_{m-1} - a_{m-2}|_v, \dots, |a_{n+1} - a_n|_v\} \\ &< \epsilon \quad \text{if } m,n > n_0, \end{aligned}$$

since $(|a_{n+1} - a_n|_v)$ converges to 0 by assumption. □

We regard a *series* as an expression of the form

$$\sum_N a_n,$$

where $a$ is a map of $N$ into $F$, and $a(n) = a_n$ for each $n \in N$. As usual, we say that a series $\Sigma_N a_n$ *converges* if its sequence of partial sums $(s_n)$ converges, where

$$s_n = a_0 + \ldots + a_n$$

for each $n \in N$. Moreover, if $(s_n)$ converges, we say that the series $\Sigma_N a_n$ converges to $\lim(s_n)$, a member of $F$, and identify the formal expression $\Sigma_N a_n$ with the field element $\lim(s_n)$, i.e.,

$$\sum_N a_n = \lim(s_n).$$

Of course, if $\Sigma_N a_n$ converges, then $\lim(a_n) = 0$. We shall show that the converse is true in the case of a nonarchimedean valuation on a complete field $\mathcal{F}$.

5.7. LEMMA. *Let $v$ be a nonarchimedean valuation on $\mathcal{F}$ which is complete with respect to $v$. Then $\Sigma_N a_n$ converges if* $\lim(a_n) = 0$.

*Proof.* Let $s_n = a_0 + \ldots + a_n$ for each $n \in N$. By assumption, $\lim(a_n) = 0$; therefore $\lim(s_{n+1} - s_n) = 0$. So, by Lemma 5.6, $(s_n)$ is a Cauchy sequence with respect to $v$. But the field $\mathcal{F}$ is complete with respect to this valuation; thus $(s_n)$ converges. We conclude that $\Sigma_N a_n$ converges. □

Next we shall prove the following lemma.

5.8. LEMMA. *Let $v$ be a valuation (archimedean or nonarchimedean) on $\mathcal{F}$ which is complete with respect to $v$. Then $\Sigma_N a_n$ converges if $\Sigma_N |a_n|_v$ converges.*

*Proof.* Let $s_n = a_0 + \ldots + a_n$ for each $n \in N$. Again, the idea is to show that $(s_n)$ is a Cauchy sequence. As in the proof of Lemma 5.6, first choose a positive real number $\epsilon$. Next observe that $(|a_0|_v + \ldots + |a_n|_v)$ is a Cauchy sequence. So there is a natural number $n_0$ such that

(5.9) $$\forall mn\, [m > n > n_0 \rightarrow |a_{n+1}|_v + \ldots + |a_m|_v < \epsilon].$$

Thus, for $m > n > n_0$,

$$|s_m - s_n|_v = |a_{n+1} + \ldots + a_m|_v \leqslant |a_{n+1}|_v + \ldots + |a_m|_v$$

by the Triangle Inequality

$$< \epsilon$$ by (5.9).

This proves that $(s_n)$ is a Cauchy sequence, so converges. We conclude that $\Sigma_N a_n$ converges. □

5.10. EXAMPLE. We now illustrate some of the above ideas and results for the case of the 2-adic valuation on $Q$ (see Example 4.1). By Lemma 5.6, the sequence $(1/2^n)$ is *not* a Cauchy sequence; thus $(1/2^n)$ does *not* converge (recall that each convergent sequence is a Cauchy sequence). On the other hand, the sequence $(2^n)$ converges; indeed, $\lim(2^n) = 0$ since

$$\lim(|2^n|_v) = \lim(e^{-n}) = 0.$$

Also

$$2 + 2^2 + \dots + 2^n = 2^{n+1} - 2$$

for each positive $n \in N$; so

$$\lim(2 + 2^2 + \dots + 2^n) = \lim(2^{n+1} - 2) = -2.$$

This means that the series $\Sigma_N\, 2^{n+1}$ converges and that $\Sigma_N\, 2^{n+1} = -2$; moreover, $\Sigma_N\, 2^n$ converges and $\Sigma_N\, 2^n = -1$.

Of course, if $\mathcal{F}$ is complete with respect to a valuation $v$, then a sequence $(a_n)$ converges iff $(a_n)$ is a Cauchy sequence. Applying Lemma 5.6, we obtain the following fact.

5.11. LEMMA. *Let $\mathcal{F}$ be complete with respect to a nonarchimedean valuation $v$. Then $(a_n)$ converges iff* $\lim(a_{n+1} - a_n) = 0$.

It is easy to see that if $(a_n)$ and $(b_n)$ converge with respect to a nonarchimedean valuation $v$, then so do $(a_n + b_n)$ and $(a_n b_n)$; indeed,

$$\lim(a_n + b_n) = \lim(a_n) + \lim(b_n),$$

$$\lim(a_n b_n) = [\lim(a_n)]\,[\lim(b_n)].$$

For this we shall need the following lemma.

5.12. LEMMA. *Let* $\lim(a_n) = a$, *where $v$ is a nonarchimedean valuation, and let $B \in R$. Then*

$$\exists n_0 \forall n\, [n > n_0 \to v(a_n) > B].$$

*Proof.* First assume that $a \neq 0$, so $v(a) \in R$. By assumption, $\lim(d(a_n, a)) = 0$, so

$$\lim(e^{-v(a_n - a)}) = 0.$$

Thus

$$\exists n_0 \forall n\, [n > n_0 \to v(a_n - a) > v(a)].$$

Since $v$ is nonarchimedean,

$$v(a_n) = v(a_n - a + a) \geqslant \min\{v(a_n - a), v(a)\} = v(a) \quad \text{if } n > n_0.$$

Thus $v(a_n) \geqslant v(a)$ if $n > n_0$. Next assume that $a = 0$. Then $\lim(d(a_n, 0)) = 0$, so

$$\lim(e^{-v(a_n)}) = 0;$$

thus

$$\forall B\, \exists n_0\, \forall n\, [n > n_0 \to v(a_n) > B].$$

In other words, $\lim(v(a_n)) = \infty$. □

We now return to our comment preceding Lemma 5.12.

5.13. LEMMA. *Let* $\lim(a_n) = a$ *and* $\lim(b_n) = b$, *where* $v$ *is a nonarchimedean valuation. Then*

$$\lim(a_n + b_n) = a + b, \qquad \lim(a_n b_n) = ab.$$

*Proof.* (i) We have

$$\begin{aligned} d(a_n + b_n, a + b) &= \exp[-v(a_n - a + b_n - b)] \\ &\leqslant \exp[-\min\{v(a_n - a), v(b_n - b)\}] \\ &= \exp[-v(c_n)], \end{aligned}$$

where for each $n \in N$, $c_n = a_n - a$ or $c_n = b_n - b$. Since

$$\lim(\exp[-v(a_n - a)]) = \lim(\exp[-v(b_n - b)]) = 0,$$

it follows that $\lim(\exp[-v(c_n)]) = 0$. Thus $\lim(d(a_n + b_n, a + b)) = 0$; so $\lim(a_n + b_n) = a + b$.

(ii)

$$d(a_n b_n, ab) = \exp[-v(a_n b_n - ab)];$$

now

$$\begin{aligned} v(a_n b_n - ab) &= v(a_n b_n - a_n b + a_n b - ab) \\ &= v(a_n[b_n - b] + b[a_n - a]) \\ &\geqslant \min\{v(a_n) + v(b_n - b), v(b) + v(a_n - a)\}. \end{aligned}$$

By Lemma 5.12, there is a $B \in R$ such that $v(a_n) > B$ if $n$ is large enough. Therefore, both $v(a_n) + v(b_n - b)$ and $v(b) + v(a_n - a)$ increase without bound as $n$ increases. Thus $\lim(d(a_n b_n, ab)) = 0$; so $\lim(a_n b_n) = ab$. □

This observation allows us to establish our next lemma.

5.14. LEMMA. *Let* $\Sigma_N a_n = a$ *and* $\Sigma_N b_n = b$. *Then*

$$\sum_N (a_n + b_n) = a + b.$$

*Proof.* The sequence of partial sums $([a_0+b_0] + \ldots + [a_n+b_n])$ converges since this sequence is the *sum* of the sequences $(a_0 + \ldots + a_n)$ and $(b_0 + \ldots + b_n)$, which converge to $\Sigma_N a_n$ and $\Sigma_N b_n$, respectively. Thus

$$\sum_N (a_n + b_n) = \sum_N a_n + \sum_N b_n = a + b. \square$$

By the *Cauchy product* of series $\Sigma_N a_n$ and $\Sigma_N b_n$ we mean the series $\Sigma_N c_n$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

for each $n \in N$ (see the definition of multiplication for Laurent series, Example 4.2). We shall prove that the Cauchy product of convergent series also converges, and that

$$\sum_N c_n = \sum_N a_n \sum_N b_n,$$

the product of the sums of the series involved.

5.15. LEMMA. *Let* $\Sigma_N a_n = a$ *and* $\Sigma_N b_n = b$. *Then* $\Sigma_N c_n = ab$, *where*

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

*for each* $n \in N$. *Here* $\mathcal{F}$ *is complete with respect to* $v$, *a nonarchimedean valuation.*

*Proof.* In a complete field with nonarchimedean valuation, a series converges iff its terms converge to 0. We shall show, first, that $\lim(c_n) = 0$. Since $\Sigma_N a_n$ and $\Sigma_N b_n$ both converge, the sets

$$\{|a_n|_v \mid n \in N\}, \qquad \{|b_n|_v \mid n \in N\}$$

are bounded, say by $B$, i.e., $|a_n|_v < B$ and $|b_n|_v < B$ for each $n \in N$. Since

$$\lim(a_n) = \lim(b_n) = 0,$$

given $\epsilon > 0$ there is a natural number $q$ such that $|a_i|_v < \epsilon/B$ and $|b_i|_v < \epsilon/B$ if $i \geqslant q$. So, for $m > 2q$,

$$|c_m|_v = |a_0 b_m + \ldots + a_m b_0|_v \leqslant \max\{|a_0 b_m|_v, \ldots, |a_m b_0|_v\}$$

$$= \max\{|a_0|_v \, |b_m|_v, \ldots, |a_m|_v \, |b_0|_v\} < \epsilon$$

since each $|a_i|_v \, |b_{m-i}|_v < B(\epsilon/B) = \epsilon$. Therefore $\lim(c_n) = 0$, so $\Sigma_N \, c_n$ converges.

To prove that $\Sigma_N \, c_n = ab$, consider the sequence of partial sums $(s_{2n})$, where

$$\begin{aligned} s_{2n} &= c_0 + \ldots + c_{2n} \\ &= \sum_{i+j=0}^{2n} a_i b_j = (a_0 + \ldots + a_n)(b_0 + \ldots + b_n) \\ &\quad + a_0(b_{n+1} + \ldots + b_{2n}) + a_1(b_{n+1} + \ldots + b_{2n-1}) + \ldots + a_{n-1} b_{n+1} \\ &\quad + b_0(a_{n+1} + \ldots + a_{2n}) + b_1(a_{n+1} + \ldots + a_{2n-1}) + \ldots + b_{n-1} a_{n+1}. \end{aligned}$$

But $\lim(b_{n+1} + \ldots + b_{n+p}) = 0$ for any $p > 0$ since

$$|b_{n+1} + \ldots + b_{n+p}|_v \leqslant \max\{|b_{n+1}|_v, \ldots, |b_{n+p}|_v\} < \epsilon/B$$

if $n > q$. The corresponding remark applies to the $a$'s; so

$$\lim(s_{2n}) = \lim(\sum_{i=0}^{n} a_i \sum_{i=0}^{n} b_i) = \lim(\sum_{i=0}^{n} a_i) \lim(\sum_{i=0}^{n} b_i) = \sum_N a_i \sum_N b_i = ab.$$

This establishes the lemma. □

Returning to our discussion of sequences, we present the following facts.

5.16. THEOREM. *Let* $\lim(a_n) = a$, *and let* $j$ *be a natural number such that* $v(a_m) = t$ *for each* $m > j$, $m \in N$. *Then* $v(a) = t$. *Here* $t \in R \cup \{\infty\}$, *and* $v$ *is a nonarchimedean valuation.*

*Proof.* By Theorem 5.1,

$$\lim(|a_n|_v) = |a|_v.$$

Thus

$$\lim(e^{-v(a_n)}) = e^{-v(a)}. \tag{5.17}$$

But $v(a_m) = t$ for each $m > j$; thus

$$\lim(e^{-v(a_n)}) = e^{-t}.$$

Therefore, from (5.17), $v(a) = t$. □

Our next lemma is a corollary of Theorem 5.16.

5.18. LEMMA. *Let $(a_n)$ be a convergent sequence, and let $j$ be a natural number such that $v(a_m) = t$ for each $m > j$, where $t \in R$. Then* $\lim(a_n) \neq 0$. *Here $v$ is a nonarchimedean valuation.*

*Proof.* Let $\lim(a_n) = a$. By Theorem 5.16, $v(a) = t$. But $t \in R$; therefore $a \neq 0$. □

We now establish the converse of Lemma 5.18; also, compare the following result to Theorem 5.1.

5.19. THEOREM. *Let* $\lim(a_n) = a$, *where* $a \neq 0$. *Then there is a natural number $j$ such that*

$$\forall m\ [m > j \rightarrow |a_m|_v = |a|_v],$$

*where the quantifier refers to $N$. Here $v$ is a nonarchimedean valuation.*

*Proof.* By (1), $|a|_v$ is positive. Since $\lim(a_n) = a$, there is a natural number $j$ such that for each $m > j$, $m \in N$,

$$|a_m - a|_v < |a|_v.$$

So, by (6), for $m > j$,

$$|a + (a_m - a)|_v = |a|_v,$$

i.e., $|a_m|_v = |a|_v$. □

Finally, we state a useful fact about convergent series.

5.20. LEMMA. *Let $v$ be a nonarchimedean valuation on $F$ which is complete with respect to $v$. Then $\Sigma_N a_n$ converges iff* $\lim(v(a_n)) = \infty$; *i.e.,*

$$\forall B\ \exists q\ \forall n\ [n > q \rightarrow v(a_n) > B]$$

*(here the first quantifier refers to $R$).*

*Proof.* By Lemma 5.7,

$$\begin{aligned} \sum_N a_n \text{ converges} \quad &\text{iff} \quad \lim(a_n) = 0 \\ &\text{iff} \quad \lim(d(a_n, 0)) = 0 \\ &\text{iff} \quad \lim(e^{-v(a_n)}) = 0 \\ &\text{iff} \quad \lim(v(a_n)) = \infty. \end{aligned}$$

Alternatively, $\Sigma_N a_n$ converges iff $\lim(|a_n|_v) = 0$. □

## 6. Hardy fields

Recall that a *real* function is a map of a subset of $R$ into $R$. Here we shall confine our attention to real functions whose domains are semi-infinite intervals, i.e., have the form $\{t \mid t > a\}$, where $a \in R$. For example, $\{(t, t^2) \mid t > -5\}$ and $\{(t, 2t + 1) \mid t > 300\}$ are real functions of this sort, whereas the functions arcsin and $\sqrt{(1 - x^2)}$ do not have this property.

Let $K$ be the set of all real functions whose domains are semi-infinite intervals. We introduce an equivalence relation $\sim$ on $K$ as follows. Let $f,g \in K$; then $f \sim g$ if there is a real number $a$ such that:

(1) $\{t \mid t > a\} \subset \operatorname{dom} f \cap \operatorname{dom} g$;

(2) $\forall t\ [t > a \rightarrow f(t) = g(t)]$.

For example,

$$\{(t, t^2) \mid t > 1\} \sim \{(t, t^2) \mid t > 20\};$$

also, $f \sim g$, where

$$f(t) = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t \leqslant 0, \end{cases}$$

$$g(t) = \begin{cases} 1 & \text{if } t > 100, \\ 0 & \text{if } t \leqslant 100. \end{cases}$$

We are interested in the resulting equivalence classes; i.e., sets of functions of the form

$$[f] = \{g \mid g \in K \wedge g \sim f\},$$

where $f \in K$. Let $H$ be the partition of $K$ obtained by collecting these equivalence classes. We define *addition* and *multiplication* on $H$ as follows:

$$[f] + [g] = [f+g], \qquad [f]\cdot[g] = [f\cdot g],$$

where the operations on the RHS are the corresponding operations on functions. Clearly the sum and product of two equivalence classes is independent of the representatives used in carrying out the definitions; i.e., the above definitions succeed in defining operations on $H$. Moreover, it is easy to verify that the resulting structure $\mathcal{H} = (H, +, \cdot, 0, 1)$ is a commutative ring with unit element. Here

$$0 = [\{(t, 0) \mid t \geqslant 0\}], \qquad 1 = [\{(t, 1) \mid t \geqslant 0\}].$$

We say that an equivalence classe, say $\psi$, is *differentiable* if there is a member of $K$, say $f$, such that:

(3) $f \in \psi$;

(4) $f$ is differenciable everywhere in its domain.

If these conditions are met, then $f' \in K$ (because the domain of $f'$ is a semi-infinite interval, by assumption). In this case, we can define the *derivative* of $\psi$ to be $[f']$, which we denote by $D\psi$ or by $\psi'$. Thus

$$D[f] = [f]' = [f']$$

in case $f$ and its derivative have the same domain. Clearly, if $f \sim g$, where $f$ and $g$ are differentiable, then $f' \sim g'$; thus $[f'] = [g']$, so $[f]' = [g]'$.

Now, by a *Hardy field* we mean a subring $\mathcal{F}$ of $\mathcal{H}$ such that:

(5) $\mathcal{F}$ is a field under the ring operations of $\mathcal{H}$;

(6) each member of $F$ is differentiable;

(7) $\forall\psi\, [\psi \in F \to \psi' \in F]$.

For example, let $F$ consist of all equivalence classes $\psi$ such that some member of $\psi$ is a *constant* function (i.e., a function whose image has exactly one member) whose domain is a semi-infinite interval. Clearly $\psi$ contains functions that are *not* constant; however, each member of $\psi$ is equivalent to the given constant function. Notice that $\psi' = 0$, the equivalence class defined above. So $\mathcal{F}$ constitutes a Hardy field.

As another example, let $[f] \in F$ iff $f$ is equivalent to a rational function (i.e., $f \sim p/q$, where $p$ and $q$ are polynomial functions). Clearly $\mathcal{F}$ is a Hardy field.

Each Hardy field can be ordered in a simple way. To see this we need the following fact.

6.1. LEMMA. *Let $[f]$ be any nonzero member of a Hardy field. Then there is a real number $b$ such that $f(t) > 0$ for each $t > b$, or else $f(t) < 0$ for each $t > b$.*

*Proof*. By assumption, $[f]$ has a multiplicative inverse, say $[g]$; also, $[f]$ is differentiable. Therefore there is a real number $b$ such that $f(t)g(t) = 1$ and $f'(t)$ exists for each $t > b$. Assume that there are real numbers $r$ and $s$, both greater than $b$, such that $f(r) > 0$ and $f(s) < 0$. But $f$ is continuous on the semi-infinite interval $\{t \mid t > b\}$; thus, by the Intermediate Value Theorem, $f$ has a zero between $r$ and $s$; so $f(t)g(t) = 0$ for some $t > b$. This contradiction establishes our lemma. □

If $f(t) > 0$ for each $t > b$, and $g \in [f]$, then there is a real number $c$ such that $g(t) > 0$ for each $t > c$. We can use this property of a Hardy field $\mathcal{F}$ to define its *positive* elements and thereby introduce an order relation on $\mathcal{F}$. We say that $[f]$ is *positive* if there is a real number $b$ such that $f(t) > 0$ for each $t > b$. Clearly $[0]$ is *not* positive; by Lemma 6.1, each nonzero field element, or its additive inverse, is positive; moreover, the sum and product of positive field elements are positive. We conclude that each Hardy field is ordered by the relation $<$ for which

$$[f] < [g] \quad \text{iff} \quad [g - f] \text{ is positive.}$$

The resulting ordered field may, or may not be, archimedean; this depends upon $\mathcal{F}$ itself. For example, let $\mathcal{F}$ be the Hardy field built around constant functions, i.e., $\psi \in F$ iff $\psi$ contains a constant function; here $\mathcal{F}$ is an archimedean field. On the other hand, let $\mathcal{F}$ be a Hardy field such that $[x] \in F$, where $x$ is the identity function $\{(t, t) \mid t \in R\}$. We claim that this Hardy field is nonarchimedean. Clearly

$$[0] < [1] < [x].$$

We shall show that $n[1] < [x]$ for each $n \in N$. Now

$$n[1] = [1] + \ldots + [1] = [\{(t, n) \mid t \in R\}] = [n].$$

Here the $n$ on the RHS is a constant function, and the $n$ on the LHS is a natural number. But $n(t) < x(t)$ for each $t > n$; thus $[x - n]$ is positive, so $n[1] < [x]$. We conclude that $\mathcal{F}$ is nonarchimedean. Notice that $[x]$ is infinite (see Section 3); it follows that $[1/x]$, the multiplicative inverse of $[x]$, is an infinitesimal.

For a more penetrating study of Hardy fields, see Bourbaki [1951] (pp. 107–126) and Robinson [1972].

## 7. The field $\mathcal{L}$

We now present a nonarchimedean field, called $\mathcal{L}$, whose field elements are generalized power series with real coefficients and *real* exponents; this field was studied by Levi–Civita late in the nineteenth century, by A. Ostrowski in the 1930's, and more recently by D. Laugwitz (see Levi–Civita [1892/93], Ostrowski [1935] and Laugwitz [1968]).

The elements of $L$ are expressions of the form

$$(7.1) \qquad a_0 t^{\nu_0} + a_1 t^{\nu_1} + a_2 t^{\nu_2} + \ldots,$$

where each $a_k, \nu_k \in R$, $\nu_0 < \nu_1 < \nu_2 < \ldots$ and $\{\nu_n \mid n \in N\}$ is unbounded. In other words, we face two sequences of real numbers, of which one is required to be strictly increasing and unbounded. More simply, we face a sequence of ordered pairs

$$(a_0, \nu_0), (a_1, \nu_1), (a_2, \nu_2), \ldots$$

whose second terms are strictly increasing and unbounded. We shall identify two sequences of this sort if each ordered pair that occurs in one, but not in the other, has first term 0. This agreement allows us to suppress any term of the series (7.1) with zero coefficient. As usual, we shall denote a series of the form (7.1) by

$$\sum_N a_k t^{\nu_k}.$$

Moreover, we shall denote

$$\sum_N 0 t^{\nu_k}$$

by 0, we shall denote the series

$$1t^0 + 0t^1 + 0t^2 + \ldots$$

by 1, and

$$t + 0t^2 + 0t^3 + \ldots$$

by $t$.

Addition is defined as the term-by-term sum of the given series, after lining up powers of $t$; i.e.,

$$\sum_N a_k t^{\nu_k} + \sum_N b_k t^{\mu_k} = \sum_N c_k t^{\lambda_k},$$

where $(\lambda_k)$ is the increasing sequence whose image is the union of the images of the sequences $(\nu_k)$ and $(\mu_k)$, i.e. $\{\nu_k \mid k \in N\} \cup \{\mu_k \mid k \in N\}$, and

$$c_k = \begin{cases} a_p + b_q & \text{if } \lambda_k = \nu_p = \mu_q, \\ a_p & \text{if } \lambda_k = \nu_p \text{ and } \lambda_k \text{ does not occur in } (\mu_i), \\ b_q & \text{if } \lambda_k = \mu_q \text{ and } \lambda_k \text{ does not occur in } (\nu_i). \end{cases}$$

For example,

$$\sum_N k t^k + \sum_N t^{2k} = \sum_N c_k t^k,$$

where

$$c_k = \begin{cases} k & \text{if } k \text{ is odd}, \\ k+1 & \text{if } k \text{ is even}. \end{cases}$$

Multiplication is defined in terms of the product of the partial sums of the given series; i.e.,

$$\left(\sum_N a_k t^{\nu_k}\right)\left(\sum_N b_k t^{\mu_k}\right) = \sum_N c_k t^{\lambda_k},$$

where $(\lambda_k)$ is the increasing sequence whose image consists of all sums of the form $\nu_i + \mu_j$, and

$$c_0 = a_0 b_0,$$

$$c_1 = \sum a_i b_j,$$

where $i$ and $j$ are chosen so that $\nu_i + \mu_j = \lambda_1$, and in general

$$c_k = \sum a_i b_j,$$

where $i$ and $j$ are chosen so that $\nu_i + \mu_j = \lambda_k$. For example,

$$\left(\sum_N t^k\right)\left(\sum_N t^{k+1}\right) = \sum_N k t^k.$$

Notice that a term of $\Sigma_N a_k t^{\nu_k}$ with zero coefficient has no impact on a sum or product involving this series; so the operations of addition and multiplication are compatible with the notion of equality for our generalized power series. It is a routine matter to verify that $\mathcal{L}$ constitutes a field under these operations. The existence of a multiplicative inverse of $\Sigma_N a_k t^{\nu_k} \neq 0$ can be established by solving the equation

$$\left(\sum_N a_k t^{\nu_k}\right)\left(\sum_N x_k t^{\mu_k}\right) = 1$$

for the unknowns $\mu_0, x_0, \mu_1, x_1, \mu_2, x_2, \ldots$ in that order. In particular,

$$\mu_0 = -\nu_0, \qquad x_0 = 1/a_0,$$

$$\mu_1 = \nu_1 - 2\nu_0, \qquad x_1 = -a_1/a_0^2.$$

It is convenient to adopt the following convention: if $\Sigma_N a_k t^{\nu_k}$ is a nonzero member of $L$, then $a_0 \neq 0$; i.e., we agree to suppress all zero terms preceding the first nonzero coefficient of a nonzero field element.

Next we wish to introduce an *order* relation on $L$. The first step is to define the positive elements of $\mathcal{L}$. We say that $\Sigma_N a_k t^{\nu_k}$ is *positive* if $a_0 > 0$ (i.e., its first nonzero coefficient is positive). Clearly, 0 is *not* positive; for each nonzero field element, either it or its additive inverse is positive; the sum and product of positive field elements are both positive. We conclude that the field $\mathcal{L}$ is ordered by the associated relation $<$, i.e., the relation defined as

$$\sum_N a_k t^{\nu_k} < \sum_N b_k t^{\mu_k} \quad \text{iff} \quad \sum_N b_k t^{\mu_k} - \sum_N a_k t^{\nu_k} \text{ is positive.}$$

We claim that the resulting ordered field is nonarchimedean. To see this, notice that $0 < t < 1$ since both $t$ and $1 - t$ are positive. But for each $n \in N$, $nt < 1$ since $1 - nt$ is positive; here $nt = \Sigma_N a_k t^k$, where $a_1 = n$ and $a_k = 0$ if $k \neq 1$. So the field element $t$ is an infinitesimal. Moreover, $\Sigma_N a_k t^{\nu_k}$ is a nonzero infinitesimal iff $\nu_0 > 0$.

In view of our last remark, we can introduce a nonarchimedean valuation $v$ on the field $\mathcal{L}$ as follows. Let $v$ be the mapping of $L$ into $R \cup \{\infty\}$ such that

$$v(0) = \infty, \qquad v\left(\sum_N a_k t^{\nu_k}\right) = \nu_0 \quad \text{if } \sum_N a_k t^{\nu_k} \neq 0$$

(so $a_0 \neq 0$, remember our convention). Clearly $v$ satisfies (1), (2) and (3) of Section 4. So $\alpha$ is a nonzero infinitesimal of the nonarchimedean field $\mathcal{L}$ iff $v(\alpha) > 0$.

Next we shall prove that $\mathcal{L}$ is complete with respect to $v$. Let $(\alpha_n)$ be any Cauchy sequence of field elements, i.e.,

$$\forall \epsilon\, \exists n_0 \forall mn\, [m,n > n_0 \to |\alpha_m - \alpha_n|_v < \epsilon].$$

It follows, in a few steps, that for each natural number $j$, $v(\alpha_m - \alpha_n) > j$ if only $m$ and $n$ are large enough (i.e., greater than some suitable $n_0$, which depends on $j$); so

$$\alpha_m - \alpha_n = \sum_N c_k t^{\lambda k},$$

where $\lambda_0 > j$. In other words, with the exception of the first few generalized power series in the sequence $(\alpha_n)$, these field elements have the same coefficient for any term whose exponent does not exceed $j$.

This fact allows us to construct the limit of the given Cauchy sequence $(\alpha_n)$. Form the sequence

$$(a_0, \nu_0), (a_1, \nu_1), (a_2, \nu_2), \ldots,$$

where $a_0 t^{\nu_0}$ is the first term of each generalized power series in $(\alpha_n)$, except for the first few; $a_1 t^{\nu_1}$ is the second term of each generalized power series in $(\alpha_n)$, except for the first few; $a_2 t^{\nu_2}$ is the third term of each generalized power series in $(\alpha_n)$, except for the first few; etc. Now let

$$\alpha = \sum_N a_k t^{\nu k}.$$

By construction, for each integer $j$, $v(\alpha - \alpha_n) > j$ if $n$ is sufficiently large; so $\lim(|\alpha - \alpha_n|_v) = 0$, thus $\lim(\alpha_n) = \alpha$. This proves that each Cauchy sequence in $\mathcal{L}$ converges (with respect to $v$). Thus $\mathcal{L}$ is complete with respect to $v$.

The exponents of a generalized power series $\Sigma_N a_k t^{\nu k}$ must satisfy two conditions:

(i) $\nu_0 < \nu_1 < \nu_2 < \ldots,$

(ii) $\{\nu_n \mid n \in N\}$ is unbounded.

These conditions are needed to ensure that generalized power series form a field; to be specific, to ensure that each nonzero power series $\Sigma_N a_k t^{\nu k}$ has a multiplicative inverse. Notice, for example, that $\Sigma_N t^{-1/k}$ does not have a multiplicative inverse.

CHAPTER 2

# NONSTANDARD ANALYSIS

## 1. The method of mathematical logic

Here, and again in Section 4, we shall establish the existence of a nonstandard model ${}^*\mathcal{R}$ of the real number system; in fact we shall prove the existence of an elementary extension of the real number system $\mathcal{R}$. In this section we shall use fundamental ideas of mathematical logic to achieve our goal; in Section 4 we shall actually construct a suitable nonstandard model of $\mathcal{R}$ by forming the ultrapower $\mathcal{R}^I/U$ of $\mathcal{R}$ with respect to a free ultrafilter $U$ over an index set $I$.

By the *real number system* one usually means the ordered field $(R, +, \cdot, <, 0, 1)$. Here, however, we shall regard the real number system in the widest possible sense, since we must include within its scope any concept or idea associated with this number system. Thus, $\mathcal{R}$ is the structure whose supporting sets include $R$, $N$, the positive real numbers, $\mathcal{P}R$, the set of all functions, etc., and which includes each concept of the real numbers as a relation of the structure. Of course, this structure has infinitely many terms. For example, we include the *upper bound* concept as a relation of the real number system $\mathcal{R}$. This means that we form the set of all ordered pairs $(S, a)$, where $S \in \mathcal{P}R$ and $a \in R$, provided that $a$ in fact is an upper bound of the set $S$. Similarly, the *least upper bound* concept is a relation of $\mathcal{R}$. We shall denote these relations by $U$ and $L$; so "$USa$" means that $a$ is an upper bound of $S$, and "$LSa$" means that $a$ is a least upper bound of $S$.

By assigning names to the relations of $\mathcal{R}$ in this manner, and utilizing the connectives of symbolic logic, we build up a language called the language of $\mathcal{R}$. This is a fragment of the informal language usually used to communicate facts about $\mathcal{R}$. To emphasize the importance of the notion of the language of a structure, we present a thorough discussion of this topic in Section 2.

In the language of $\mathcal{R}$ we can express the *Completeness Theorem* of the real number system as follows:

$$(1.1) \qquad \forall S\, [S \neq \emptyset \wedge \exists x\, USx \rightarrow \exists y\, LSy].$$

Now the *Compactness Theorem* of mathematical logic asserts that a set of statements (within a certain, well-defined language) has a model if each of its finite subsets has a model. An acceptable set of statements is obtained by writing down all statements, such as (1.1), that are true for $\mathcal{R}$, and belong to the fragment of the informal language of $\mathcal{R}$ characterized in Section 2. This is why we include all possible concepts of the real numbers in $\mathcal{R}$. Moreover, we shall include in our postulate set the statements

$$\omega \in N, \omega > 0, \omega > 1, \omega > 2, \ldots,$$

where $\omega$ is an uninterpreted placeholder (i.e., a free variable); we choose $\omega$ to be a symbol that does *not* occur among our other postulates.

Notice that each finite subset of this postulate set has a model, e.g., $\mathcal{R}$ itself. By assumption, each postulate of the first sort is true for $\mathcal{R}$. Moreover, any finite number of postulates of the second sort can be satisfied in $\mathcal{R}$ by interpreting $\omega$ as the smallest natural number greater than each natural number occurring in the RHS of these postulates. Thus, by the Compactness Theorem, our postulate set has a model; we seize on one of the models of our postulate set and call it ${}^*\mathcal{R}$.

It must be emphasized that ${}^*\mathcal{R}$ is a structure patterned on $\mathcal{R}$; in general, each term of ${}^*\mathcal{R}$ corresponds to a unique term of $\mathcal{R}$ (for exceptions to this statement, see Section 2). We indicate this by denoting a term of ${}^*\mathcal{R}$ by starring the corresponding term of $\mathcal{R}$, if there is one. So ${}^*R$ corresponds to $R$, and ${}^*N$ corresponds to $N$. This means that the *real numbers* of ${}^*\mathcal{R}$ are the members of ${}^*R$, and that the *natural numbers* of ${}^*\mathcal{R}$ are the members of ${}^*N$. For each $a \in R$, "$a < a + 1$" is in our postulate set; thus ${}^*R$ contains a number identified with $a$, which we take to be $a$ itself.

The point is that ${}^*\mathcal{R}$ is a structure of the same type as $\mathcal{R}$, indeed is an extension of $\mathcal{R}$, such that:

(1) ${}^*R$ is a proper superset of $R$.

(2) Each statement that is true for $\mathcal{R}$ is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$.

These structures are different because ${}^*\mathcal{R}$ possesses a number, corresponding to $\omega$, that is greater than each natural number, so is greater than each real number (in the usual sense); for simplicity, we shall denote this number by "$\omega$". Of course, (2) is a consequence of our choice of ${}^*\mathcal{R}$ as a model of a postulate set that includes all statements that are true for $\mathcal{R}$.

Notice that the converse of (2) follows from (2) itself. This establishes the vitally important *Transfer Theorem*, which we now state; here ${}^*A$ represents the interpretation of $A$ in ${}^*\mathcal{R}$, i.e., ${}^*A$ is the statement of ${}^*\mathcal{R}$ obtained from $A$ by starring its relations and starring each occurrence in $A$ of a member of a supporting set of $\mathcal{R}$.

1.2. TRANSFER THEOREM. *Let A be any statement in the language of* $\mathcal{R}$. *Then* ${}^*A$ *is true for* ${}^*\mathcal{R}$ *iff A is true for* $\mathcal{R}$.

The Transfer Theorem is extremely powerful since it allows us to utilize our knowledge of $\mathcal{R}$ in studying ${}^*\mathcal{R}$. Bear in mind the restriction on $A$; $A$ must be a statement in the language of $\mathcal{R}$ (see Section 2). Also, we must take care when verbalizing a statement ${}^*A$ in the language of ${}^*\mathcal{R}$. The fact is that a supporting set, or relation, of ${}^*\mathcal{R}$ need not have quite the same significance as the corresponding supporting set, or relation, of $\mathcal{R}$.

To illustrate the pitfalls, consider the Completeness Theorem. Now (1.1) is true for $\mathcal{R}$; so (1.1) is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$. at first glance, this appears to be a paradox in view of the following argument, which purports to demonstrate that the Completeness Theorem is *not* true for ${}^*\mathcal{R}$:

"Certainly, ${}^*\mathcal{R}$ possesses infinitesimals (since the multiplicative inverse of an infinite number is an infinitesimal). Let $S$ be the set of all infinitesimals of ${}^*\mathcal{R}$. Then $S$ is nonempty and has an upper bound; indeed, each positive $a \in R$ is an upper bound of $S$. Suppose, for the moment, that $S$ has a least upper bound, say $t$. Either $t$ is a positive infinitesimal or $t$ is greater than some positive real number $a$. In the latter case, $a$ itself is an upper bound of $S$, so $t$ is not its least upper bound. In the former case, $2t \in S$ and $t \,{}^*\!< 2t$; so $t$ is not an upper bound of $S$. We conclude that $S$ does not have a least upper bound; so the Completeness Theorem is not true for ${}^*\mathcal{R}$."

The fallacy in this argument consists in failing to interpret (1.1) in ${}^*\mathcal{R}$. After all, what we mean by the *Completeness Theorem for* ${}^*\mathcal{R}$ is the interpretation of (1.1) in ${}^*\mathcal{R}$. We do not mean the following: "Each nonempty subset of ${}^*R$ that has an upper bound, has a least upper bound". To see the distinction we must write out the Completeness Theorem for $\mathcal{R}$ in detail, avoiding the streamlined notation in which quantifiers refer to specific supporting sets. Imagine, then, that we have equipped both $\mathcal{R}$ and ${}^*\mathcal{R}$ with a basic set, the union of the supporting sets of the structure involved. The Completeness Theorem for $\mathcal{R}$ is

$$(1.3)\qquad \forall z\,[z \in \mathcal{P}R \rightarrow (z \neq \emptyset \wedge \exists x\,[x \in R \wedge Uzx] \rightarrow \exists y\,[y \in R \wedge Lzy])].$$

Interpreting this statement in ${}^*\mathcal{R}$ yields

$$(1.4)\qquad \forall z\,[z \in {}^*(\mathcal{P}R) \rightarrow (z \neq \emptyset \wedge \exists x\,[x \in {}^*R \wedge {}^*Uzx] \rightarrow \exists y\,[y \in {}^*R \wedge {}^*Lzy])].$$

Indeed (1.4) is true for ${}^*\mathcal{R}$; notice that (1.4) does not assert that each nonempty subset of ${}^*\mathcal{R}$ that has an upper bound, has a least upper bound. Instead, (1.4) asserts that each nonempty set contained in ${}^*(\mathcal{P}R)$ that has an upper bound, has a least upper bound. The fact is that ${}^*(\mathcal{P}R) \neq \mathcal{P}({}^*R)$; i.e., some subsets of

$^*R$ are not contained in $^*(\mathcal{P}R)$. In particular, the above paradox demonstrates that $S \notin {}^*(\mathcal{P}R)$, where $S$ is the set of all infinitesimals of $^*\mathcal{R}$.

Fundamentally, each relation, or concept, of a structure is characterized by a set. Indeed, the truth or falsity of a statement in $\mathcal{R}$ or $^*\mathcal{R}$ depends upon the sets that represent the concepts involved in the statement. Generally, the set representing a concept in $^*\mathcal{R}$ is not the set that represents it in $\mathcal{R}$. More to the point, the set that represents a concept in $^*\mathcal{R}$ generally cannot be verbalized in the same direct and simple fashion as for $\mathcal{R}$. For example, the set of all subsets of $R$ is *not* represented in $^*\mathcal{R}$ by the set of all subsets of $^*R$; rather, it is represented by $^*(\mathcal{P}R)$, a certain set of subsets of $^*R$.

The idea of interpreting a statement in a structure can be illustrated by considering the statement

$$\forall x\,[x \neq 0 \rightarrow \exists y\,[xy = 1]].$$

We are accustomed to interpreting this statement in several number systems, e.g. the real number system, the rational number system, and the system of integers. To determine its truth value in a particular number system, we consider the operation of multiplication of that number system and the number set involved. We arrive at the conclusion that this statement is true for the real number system and for the rational number system, but is false for the system of integers. Notice that we have interpreted a statement in a number system, by interpreting the concepts involved in the statement, and have reached a decision regarding its truth or falsity in that structure.

Returning to our description of the real number system $\mathcal{R}$, we point out that each real function can be represented by a relation on $R$. This is due to the fact that each function is characterized by a set of ordered pairs; so, a function is a binary relation, i.e., a subset of $R \times R$. Moreover, any binary relation on $R$ can be regarded as a function provided that no two of its members have the same first term. Our point, here, is that specific functions can be terms of $\mathcal{R}$. Let $f$ be a function, and suppose that the corresponding relation, which we also denote by $f$, is a term of $\mathcal{R}$. We mention that the mathematical statement $f(a) = b$ is rendered in the language of $\mathcal{R}$ by $(a, b) \in f$, which is sometimes abbreviated by writing $fab$.

As stated earlier, it is often convenient to use the language of operations; this we shall freely do. To illustrate, consider the statement

$$(1.5) \qquad f(x) - f(y) < h.$$

This can be expressed in terms of relations as follows:

$$(1.6) \qquad \exists uvw\,[(x, u) \in f \wedge (y, v) \in f \wedge (w, v, u) \in + \wedge (w, h) \in <].$$

Comparing the readability of these statements, it is evident that we should utilize the language of functions in this case (and in many other cases).

As another step toward improved readability, we shall frequently suppress the stars that are part of the names of some relations of $^*\mathcal{R}$ – provided that it is clear from the context that we are referring to $^*\mathcal{R}$, not $\mathcal{R}$.

In the remainder of this section we shall present a few basic facts about $^*\mathcal{R}$, enough to understand the nonstandard analysis used in this book. We mention, first of all, that the substructure of $^*\mathcal{R}$

$$(^*R, +, \cdot, <, 0, 1),$$

where *'s have generally been suppressed, is a nonarchimedean field. Accordingly, we shall speak of the *infinitely large* elements of $^*R$, the *infinitesimals* of $^*R$, and the *finite* members of $^*R$ (see Section 1.3).

Here is an important equivalence relation on $^*R$. Let $a,b \in {}^*R$; we shall write $a \simeq b$ (read "*a is infinitely close to b*") provided that $a - b$ is an infinitesimal. Clearly 0 is an infinitesimal, $-\epsilon$ is an infinitesimal if $\epsilon$ is an infinitesimal, and the sum of two infinitesimals is an infinitesimal (see Section 1.3). It follows that the binary relation $\simeq$ is an equivalence relation on $^*R$.

The following is a basic fact about the finite members of $^*R$.

1.7. FUNDAMENTAL THEOREM ABOUT FINITE NUMBERS. *Each finite number is infinitely close to a unique member of $R$.*

We leave the proof to the reader; see Robinson [1966] or Lightstone [1972].

In other words, each finite $a \in {}^*R$ can be represented uniquely as a sum of the form $r + \epsilon$, where $r \in R$ and $\epsilon \simeq 0$. Recall that we have agreed to identify the image in $^*R$ of each member of $R$ with the member of $R$ involved.

As an illustration of the Fundamental Theorem about Finite Numbers, we mention that in $^*\mathcal{R}$, each interval of infinitesimal length contains at most one number in $R$. For example, the open interval $(t - \epsilon, t + \epsilon)$, where $t \in R$ and $\epsilon$ is a positive infinitesimal, contains exactly one number in $R$, $t$ itself; the open interval $(\epsilon, 2\epsilon)$ contains no number in $R$; the open interval $(\omega, \omega + \epsilon)$, where $\omega$ is infinite, contains no number in $R$.

If $a = r + \epsilon$, as in our theorem, then $r$ is called the *standard* part of $a$, and is denoted by $^0a$. Indeed, each member of $R$ is said to be *standard*.

Consider again the fact that each statement true for $\mathcal{R}$ is true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$ (see (2) above). As we have pointed out, it is crucial to realize that a concept of $^*\mathcal{R}$ may differ in a subtle manner from the corresponding concept of $\mathcal{R}$. A useful tool for studying this distinction is the notion of an *internal* entity of $^*\mathcal{R}$ and the complementary notion of an *external* entity. Now a concept of $\mathcal{R}$ such as the notion of the set of all subsets of $R$, in ${}^*\mathcal{R}$ is merely a set of certain subsets of ${}^*R$. Each member of ${}^*(\mathcal{P}R)$ is said to be an *internal* subset of ${}^*R$; any other subset of ${}^*R$ is called an *external* subset of ${}^*R$. This terminology applies to each supporting set of ${}^*\mathcal{R}$. For example, the set of all functions is a supporting set of $\mathcal{R}$; in ${}^*\mathcal{R}$ the corresponding supporting set is merely a set of certain functions (i.e., maps of an internal subset of ${}^*R$ into ${}^*R$). Each function in this set is said to be an *internal* function; any other map of a subset of ${}^*R$ into ${}^*R$ is called an *external* function. Similarly we shall speak in ${}^*\mathcal{R}$ of its *internal* or *external* sequences.

The importance of internal versus external entities rests on the fact that each statement that is true for $\mathcal{R}$ is true for ${}^*\mathcal{R}$ provided its quantifiers are restricted to the internal entities of ${}^*\mathcal{R}$.

Let us, once again, illustrate this vital point. Now Peano's induction postulate

$$(1.8) \qquad \forall S\,[1 \in S \wedge \forall x\,[x \in S \rightarrow x + 1 \in S] \rightarrow S = N]$$

is certainly true for $\mathcal{R}$. Therefore (1.8) is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$; i.e.,

$$(1.9) \qquad \forall S\,[1 \in S \wedge \forall x\,[x \in S \rightarrow x + 1 \in S] \rightarrow S = {}^*N]$$

is true for ${}^*\mathcal{R}$ provided that we read "$\forall S$" as "for each internal subset $S$ of ${}^*N$". Thus (1.9) asserts that there is just one internal subset of ${}^*N$, namely ${}^*N$ itself, such that both 1 is in the subset and the successor of each member of the subset is in the subset. Of course, $N$ has both these properties; we also know that $N \neq {}^*N$, so we must conclude that $N$ is an *external* subset of ${}^*N$.

## 2. The languages of $\mathcal{R}$ and ${}^*\mathcal{R}$

Here we shall spell out the fragment of the informal language of a structure which we used in Section 1. We shall follow the procedure used to define the language of the predicate calculus, a basic notion of mathematical logic. For each structure, then, the resulting language follows a common pattern. By restricting our attention to this fragment of the informal language of a structure, we can take advantage of powerful metatheorems of mathematical logic (i.e. theorems *about* the language, e.g. the Transfer Theorem 1.2 and Łoś Lemma 4.1). Moreover, there is a well-defined procedure for determining the truth or falsity, in the structure involved, of a statement in the restricted language.

Turning to the real number system $\mathcal{R}$, we regard numbers, tuples of numbers, and sets of numbers as objects; more generally, the members of the supporting sets of $\mathcal{R}$ are called objects. These, together with the relations of $\mathcal{R}$, generate statements such as

$$3 \in N, \qquad (2, 5) \in <, \qquad (2, 5, 7) \in +,$$

which are true. On the other hand, we can formulate the statements

$$-3 \in N, \qquad (5, 2) \in <, \qquad (2, 5, 6) \in +,$$

which are false. More usually, some of these statements are expressed by:

$$2 < 5, \qquad 2 + 5 = 7, \qquad 5 < 2, \qquad 2 + 5 = 6.$$

Each statement of the form $\alpha \in T$, where $\alpha$ is a tuple of objects of $\mathcal{R}$ (i.e. members of its supporting sets) and $T$ is a relation of $\mathcal{R}$, is called an *atomic* statement of $\mathcal{R}$. The statement $\alpha \in T$ is true if the tuple $\alpha$ is a member of the set $T$; otherwise $\alpha \in T$ is false. Each atomic statement of $\mathcal{R}$ has a unique truth value.

We build on the atomic statements of $\mathcal{R}$ by utilizing the connectives of symbolic logic. Let $p$ and $q$ be any statements of our language, either atomic statements introduced above or more complicated statements built up from atomic statements by means of our connectives. Then we say that each of the following is a statement of $\mathcal{R}$:

$\neg p$ (not $p$),

$p \vee q$ ($p$ or $q$),

$p \wedge q$ ($p$ and $q$),

$p \rightarrow q$ (if $p$, then $q$),

$p \leftrightarrow q$ ($p$ iff $q$).

Thus $\neg p$ is true if $p$ is false, and is false if $p$ is true; $p \vee q$ is false just in case both $p$ and $q$ are false; $p \wedge q$ is true just in case both $p$ and $q$ are true; $p \rightarrow q$ is false just in case $p$ is true and $q$ is false; $p \leftrightarrow q$ is true if $p$ and $q$ have the same truth value, and is false if $p$ and $q$ have different truth values.

Continuing our definition, let $P(x)$ be a statement form, i.e. an expression involving a placeholder, here $x$, such that replacing $x$ by an object yields a statement. Then we say that $\forall x\ [P(x)]$, which we sometimes write more simply as $\forall x P(x)$, is a statement of $\mathcal{R}$; this statement is true for $\mathcal{R}$ provided that each statement of the form $P(a)$, where $a \in A$ and $A$ is a specified supporting set of $\mathcal{R}$, is true for $\mathcal{R}$. Thus each quantifier refers to some supporting set of $\mathcal{R}$; as we have mentioned earlier, this is usually done typographically. Similarly, $\exists x\ [P(x)]$ is a statement of $\mathcal{R}$; this statement is true for $\mathcal{R}$ provided that at least one statement of the form $P(a)$, where $a \in A$ and $A$ is the supporting set of $\mathcal{R}$ referred to by the quantifier, is true for $\mathcal{R}$.

We mention that each statement in the language of $\mathcal{R}$ must be of finite length; i.e., each statement may contain only a finite number of instances of connectives (so only a finite number of instances of atomic statements).

The language of ${}^*\mathcal{R}$, indeed of any structure, is defined by following the same procedure as in the case of $\mathcal{R}$. First we define the *atomic* statements of the language; thus each expression of the form $\alpha \in T$, where $\alpha$ is a tuple of objects of ${}^*\mathcal{R}$ (i.e., members of its supporting sets) and $T$ is a relation of ${}^*\mathcal{R}$, is called an *atomic* statement of ${}^*\mathcal{R}$. The statement $\alpha \in T$ is *true* if the tuple $\alpha$ is a member of the set $T$; otherwise $\alpha \in T$ is said to be *false*. Just as for the language of $\mathcal{R}$, we build on the atomic statements of ${}^*\mathcal{R}$ by utilizing the connectives of symbolic logic. Moreover, we define the truth value of each of the resulting statements in terms of the truth values of its components, just as in the case of the language of $\mathcal{R}$.

As we have suggested, certain relations of ${}^*\mathcal{R}$ correspond to relations of $\mathcal{R}$; each relation of this sort is denoted by starring the corresponding relation of $\mathcal{R}$. Each statement in the language of ${}^*\mathcal{R}$ which involves only relations that correspond to relations of $\mathcal{R}$ in this way yields a statement in the language of $\mathcal{R}$ by merely removing all *'s and interpreting all quantifiers appropriately (i.e., as referring to a supporting set of $\mathcal{R}$, not ${}^*\mathcal{R}$).

There are two sorts of statements in the language of ${}^*\mathcal{R}$: those that yield a statement in the language of $\mathcal{R}$, as just described, and those that do not yield a statement in the language of $\mathcal{R}$. A statement of the latter sort will involve a relation of ${}^*\mathcal{R}$ that does not correspond to any relation of $\mathcal{R}$, or it will involve an external object of ${}^*\mathcal{R}$, or it will involve an internal object that is *not* rooted in an object of $\mathcal{R}$. For example, the equivalence relation $\simeq$, introduced in Section 1, is a relation of ${}^*\mathcal{R}$ but not of $\mathcal{R}$; the set of all infinitesimals, the set of all infinite numbers, and the set of all finite numbers form unary relations of ${}^*\mathcal{R}$ but not of $\mathcal{R}$. We point out that the Fundamental Theorem about Finite Numbers 1.7 is a statement of ${}^*\mathcal{R}$ which does not reduce to a statement of $\mathcal{R}$.

This classification of the statements of ${}^*\mathcal{R}$ into two classes is of fundamental importance; for example, see the Ultrapower Theorem 4.2. To sharpen this concept, we shall refer to the *internal* language of ${}^*\mathcal{R}$ and to the *external* language of ${}^*\mathcal{R}$. A statement in the language of ${}^*\mathcal{R}$ is in the internal language (and is called an *internal* statement of ${}^*\mathcal{R}$) provided that it is the interpretation in ${}^*\mathcal{R}$ of some statement in the language of $\mathcal{R}$. The *external* language of ${}^*\mathcal{R}$ consists of all other statements (called *external* statements of ${}^*\mathcal{R}$) in the language of ${}^*\mathcal{R}$.

For example, any statement of ${}^*\mathcal{R}$ that involves $N$, $R$, $\{\epsilon \mid \epsilon \simeq 0\}$ or any other external object of ${}^*\mathcal{R}$ is an external statement of ${}^*\mathcal{R}$.

## 3. Filters

The purpose of this section is to establish the existence of a free ultrafilter over a denumerable index set.

We begin by presenting the notion of a *filter* over an index set $I$. Let $F$ be a collection of subsets of $I$ such that:

(1) $\forall AB\,[A \in F \wedge A \subset B \subset I \rightarrow B \in F]$ (the *superset* requirement);

(2) $\forall AB\,[A,B \in F \rightarrow A \cap B \in F]$ (the *intersection* requirement);

(3) $\emptyset \notin F$ (the *empty set* requirement);

then $F$ is said to be a *filter* over $I$.

Here are some examples.

3.1. EXAMPLE. Take $\{1, 2\}$ as the index set, and let $F = \{\{1\}, \{1, 2\}\}$. Then $F$ is a filter over $\{1, 2\}$.

3.2. EXAMPLE. Take $N = \{0, 1, 2, ...\}$ as the index set, and let $F = \{A \mid A \subset N \wedge 0 \in N\}$. Then $F$ is a filter over $N$.

Notice that there is just one filter over the empty set, namely the empty set. For this reason, many authors require an index set to be nonempty. More generally, the empty set is a filter over any index set.

The filters of our examples have the property that $\bigcap_{A \in F} A \neq \emptyset$; indeed, some member of the index set is in each element of the filter. Any filter with this property is said to be *fixed*. A filter which is not fixed is said to be *free*.

3.3. EXAMPLE. Take $N$ as the index set, and let $A \in F$ iff $A$ is a cofinite subset of $N$, i.e., $N - A$ is finite. It is easy to verify that $F$ is a filter over $N$. Moreover, $F$ is free. We must show that no member of the index set is in each element of the filter. Let $n \in N$. Then $A \in F$, where $A = N - \{n\}$. But $n \notin A$. This proves that $F$ is free.

3.4. EXAMPLE. Let $I$ be any infinite set, and let $A$ be any infinite subset of $I$. Define $F$ as follows. Let $S \in F$ iff $S \subset I$ and $S$ contains infinitely many members of $A$. Certainly $F$ satisfies the superset requirement and the empty set requirement; however, $F$ does not satisfy the intersection requirement (e.g., there are two disjoint subsets of $I$ each of which contains infinitely many members of $A$). So $F$ is *not* a filter over $I$.

Next we want to characterize a filter $F$ over $I$, a non-empty index set, which cannot be extended to a bigger filter over $I$; i.e., if $A$ is a subset of $I$

such that $F \cup \{A\}$ is a filter, then $A \in F$. This is the notion of a *maximal* filter over $I$. It turns out that a filter $F$ over $I$, a nonempty index set, is maximal provided that $F$ satisfies the following condition:

(3′) $\forall A\, [A \subset I \rightarrow A \in F \veebar I - A \in F]$ (the *ultrafilter* requirement).

Any filter that satisfies the ultrafilter requirement is said to be an *ultrafilter*. Notice our use of the "exclusive or" in (3′).

We mention that the empty set requirement is deducible from the superset requirement, the intersection requirement and the ultrafilter requirement, provided that the index set is nonempty. Accordingly, we can characterize an ultrafilter over a nonempty index set as follows.

3.5. THEOREM. *Let $I$ be any nonempty set, and let $U$ be a collection of subsets of $I$ such that*:

(1) $\forall AB\, [A \in U \wedge A \subset B \subset I \rightarrow B \in U]$;

(2) $\forall AB\, [A, B \in U \rightarrow A \cap B \in U]$;

(3′) $\forall A\, [A \subset I \rightarrow A \in U \veebar I - A \in U]$.

*Then $U$ is an ultrafilter over $I$.*

*Proof.* Since $I$ is nonempty, it follows from (3′) that $U$ is nonempty. Indeed, let $x \in I$; then $\{x\} \subset I$, so either $\{x\} \in U$ or $I - \{x\} \in U$ (but not both). It may be that $I - \{x\} = \emptyset$; even so, $U$ possesses a member, $\{x\}$ or $I - \{x\}$. By (1), each superset (in $I$) of a member of $U$ is also in $U$; therefore $I \in U$. So, by (3′), $\emptyset \notin U$. This proves that $U$ meets the empty set requirement, so $U$ is a filter over $I$. This completes our proof. □

The filter of Example 3.2 is an ultrafilter over $N$; this is a fixed ultrafilter since 0 is in each element of the filter. The filter of Example 2.3 is not an ultrafilter; for example, neither $\{1, 3, 5, \ldots\}$ nor $\{0, 2, 4, \ldots\}$ is in this filter.

We have not yet presented an example of a free ultrafilter. The difficulty is due to the fact that the index set of a free ultrafilter must be infinite.

3.6. LEMMA. *Each filter that possesses a finite subset of the index set is fixed.*

*Proof.* Let $F$ be a free filter over an index set $I$, and let $T$ be a finite subset of $I$ such that $T \in F$. Since $F$ is free, corresponding to each member $t$ of $T$ there is a member $A_t$ of $F$ such that $t \notin A_t$. By the intersection requirement, $A_t \cap T \in F$ for each $t \in T$. So $\bigcap_{t \in T}(A_t \cap T) \in F$, again by the intersection requirement. But $\bigcap_{t \in T}(A_t \cap T)$ is the empty set. This contradicts the empty set requirement. We conclude that no finite subset of $I$ is in $F$. So each filter that possesses a finite subset of the index set is fixed. □

To prove the existence of a free ultrafilter we shall make use of Zorn's lemma, which we now recall.

3.7. ZORN'S LEMMA. *Let $P$ be a nonempty partially ordered set for which each simply ordered subset has an upper bound. Then $P$ has a maximal element.*

We explain that a *maximal* element of $P$ is a member of $P$ which precedes no other member of $P$ (with respect to the given partial ordering).

Throughout the remainder of this section, we shall restrict our attention to *nonempty* index sets.

It is clear that each ultrafilter is maximal over its index set. It is not so clear that each maximal filter over a nonempty index set $I$ is an ultrafilter. Our plan is to prove that each filter over $I$ can be extended to a maximal filter over $I$ (i.e., is a subset of a maximal filter over $I$). Next, we shall prove that if $F$ is maximal, then $F$ is an ultrafilter. Finally, we shall prove that if the index set $I$ is infinite, then the filter of cofinite subsets of $I$ can be extended to $U$, a maximal filter over $I$. Since $U$ is maximal $U$ is an ultrafilter; since $U$ contains the filter of cofinite subsets of $I$, $U$ is free.

To begin this program, let $I$ be any nonempty set, let $F_0$ be any filter over $I$, and let $P$ be the set of all filters $F$ over $I$ such that $F_0 \subset F$. The subset relation $\subset$ is a partial ordering on $P$. Moreover, each simply ordered subset of $P$ has an upper bound in $P$, namely its union (we shall prove this in a moment). Therefore, by Zorn's Lemma, $P$ has a maximal element, say $U$.

We shall now prove our statements.

3.8. LEMMA. *Let $S$ be any simply ordered subset of $P$, and let $F = \bigcup S$. Then $F \in P$.*

*Proof.* By construction, $F$ is a superset of $F_0$. We must show that $F$ is a filter over $I$.

(i) Let $A \in F$, and let $A \subset B \subset I$. There is a member of $S$, say $F_i$, such that $A \in F_i$; thus $B \in F_i$ (since $F_i$ is a filter over $I$). Therefore $B \in F$.

(ii) Let $A,B \in F$. There are members of $S$, say $F_i$ and $F_j$, such that $A \in F_i$ and $B \in F_j$. Since $S$ is simply ordered, $F_i \subset F_j$ or $F_j \subset F_i$. So $A,B \in F_k$, where $k \in \{i,j\}$. Therefore $A \cap B \in F_k$; so $A \cap B \in F$.

(iii) If $\emptyset \in F$, there is a member of $S$, say $F_i$, such that $\emptyset \in F_i$. This is impossible since $F_i$ is a filter. We conclude that $\emptyset \notin F$. This completes our proof that $F$ is a filter over $I$; so $F \in P$. □

By Zorn's Lemma, $P$ has a maximal element, say $U$; by construction, $U$ is a superset of $F_0$ and is a filter over $I$. This establishes the following fact.

3.9. LEMMA. *Each filter over $I$, a nonempty set, can be extended to a maximal filter over $I$.*

Next we want to prove that each maximal filter over $I$, a nonempty set, is an ultrafilter over $I$. Let $U$ be maximal and suppose that $U$ is not an ultrafilter. Then there is a nonempty subset $A$ of $I$ such that $A \notin U$ and $I - A \notin U$. Our plan is to regard $A$ as an index set, and to construct a filter over $A$ from the sets in $U$. Delete from each member of $U$ all members of $I - A$; this yields a collection of subsets of $A$ which we call $F$. The idea is to use $F$ to extend $U$ to a bigger filter over $I$. First we shall show that $F$ is a filter over $A$.

3.10. LEMMA. *$F$ is a filter over $A$.*

*Proof.* (i) Let $A_1 \in F$, and let $A_1 \subset A_2 \subset A$. There is a subset of $I - A$, say $B$, such that $A_1 \cup B \in U$. But $A_1 \cup B \subset A_2 \cup B$, so $A_2 \cup B \in U$. Thus $A_2 \in F$.

(ii) Let $A_1, A_2 \in F$. By assumption, there are subsets of $I - A$, say $B_1$ and $B_2$, such that $A_1 \cup B_1 \in U$ and $A_2 \cup B_2 \in U$. By the intersection requirement,

$$(A_1 \cup B_1) \cap (A_2 \cup B_2) \in U,$$

i.e.,

$$(A_1 \cap A_2) \cup (B_1 \cap B_2) \in U,$$

so $A_1 \cap A_2 \in F$.

(iii) If $\emptyset \in F$, there is a subset of $I - A$, say $B$, such that $B \in U$. By the superset requirement, $I - A \in U$. But $I - A \notin U$; we conclude that $\emptyset \notin F$. This proves that $F$ is a filter over $A$. □

We propose to use $F$ to extend $U$ to a bigger filter over $I$. Adjoin to $U$ each member of $F$, together with each of its supersets in $I$. The resulting collection of subsets of $I$, which we shall call $U'$, is a filter over $I$.

3.11. LEMMA. *$U'$ is a filter over $I$.*

*Proof.* The superset requirement is met by construction. Let $C, D \in U'$. Now $C \cap D \in U'$ if both $C$ and $D$ are in $U$ or if both are in $F$, more generally, if both $C$ and $D$ are supersets of elements of $F$. Assume that $C \in U$ and $D \in F$ (this covers the case that $D$ is a superset of an element of $F$). Then there is a subset of $I - A$, say $B_1$, and a subset of $I$, say $A_1$, such that $C = A_1 \cup B_1$.

Thus

$$C \cap D = (A_1 \cup B_1) \cap D = A_1 \cap D \in F$$

(since $F$ is a filter). So $C \cap D \in U'$. This shows that $U'$ meets the intersection requirement. Since $\emptyset \notin F$ and $\emptyset \notin U$, $\emptyset$ is not in $U'$. We conclude that $U'$ is a filter over $I$. □

By construction, $U \subset U'$; indeed, $U$ is a proper subset of $U'$. Thus $U$ is not a maximal filter over $I$. This contradiction proves that $U$ is an ultrafilter.

3.12. LEMMA. *Each maximal filter over I, a nonempty set, is an ultrafilter.*

Lemma 3.12 establishes our claim that the ultrafilter requirement is a criterion for a maximal filter over a nonempty index set.

Finally we shall establish the existence of a free ultrafilter. Take $I$ infinite; by Lemma 3.9, the filter of cofinite subsets of $I$ can be extended to a maximal filter over $I$, say $U$. But the filter of cofinite subsets is free; thus $U$ is free. Moreover, by Lemma 3.12, $U$ is an ultrafilter. We conclude that $U$ is a free ultrafilter. This establishes the following fact.

3.13. THEOREM. *There is a free ultrafilter over I, provided that I is infinite.*

## 4. The ultrapower construction

In Section 1 we showed that there is an extension of the real number system, which we called $^*\mathcal{R}$, that possesses infinitesimals and possesses each property of the real number system. Here we present another method of establishing this fact.

Let $U$ be a free ultrafilter over a denumerable index set $I$. We shall construct a model of the postulate set of Section 1, which we shall again call $^*\mathcal{R}$, by utilizing $\mathcal{R}$, the index set $I$ and the free ultrafilter $U$. By $^*R$ we shall mean the set $R^I$ of all maps of $I$ into $R$; this is the number set of our structure $^*\mathcal{R}$. Here each number is a map of $I$ into $R$. Each supporting set of $\mathcal{R}$ yields a supporting set of $^*\mathcal{R}$ in exactly the same way. For example, suppose that $N$, Seq, $\mathcal{P}R$ and $F$ are supporting sets of $\mathcal{R}$, where $N$ is the set of all natural numbers, Seq is the set of all sequences (of standard numbers), $\mathcal{P}R$ is the set of all subsets of $R$, and $F$ is the set of all functions on $R$ (i.e., all maps of a subset of $R$ into $R$). Then in $^*\mathcal{R}$ the terms *real number*, *natural number*, *sequence*, *subset* and *function* have the following meaning:

| | |
|---|---|
| real number: | map of $I$ into $R$; |
| natural number: | map of $I$ into $N$; |
| sequence: | map of $I$ into Seq; |
| subset: | map of $I$ into $\mathcal{P}R$; |
| function: | map of $I$ into $F$. |

Similarly, each supporting set of $\mathcal{R}$ is represented in $^*\mathcal{R}$ by the set of all maps of $I$ into that supporting set. We shall refer to members of the supporting sets of $\mathcal{R}$ as *standard* objects.

Next we construct the relations of $^*\mathcal{R}$; here we shall use our ultrafilter $U$. Each relation of $^*\mathcal{R}$ is obtained from a relation of $\mathcal{R}$ by the following procedure. Let $T$ be a relation of $\mathcal{R}$; then $^*T$, the corresponding relation of $^*\mathcal{R}$, is defined by

$$(f_1, ..., f_n) \in {}^*T \quad \text{iff} \quad \{\nu \in I \mid (f_1(\nu), ..., f_n(\nu)) \in T\} \in U.$$

The underlying idea is that each relation of a structure is a set of tuples; we decide which tuples are members of $^*T$ by asking whether the subset of $I$ that yields members of $T$, as described in the definition, is a member of the ultrafilter. Of course, a relation can connect (i.e., relate) objects from different supporting sets of the structure; thus the maps $f_1, ..., f_n$ mentioned above may be drawn from different supporting sets of $^*\mathcal{R}$. For example, let VS be the relation of $\mathcal{R}$ that expresses the value of a sequence at a natural number; here a typical member of VS is a tuple $(\alpha, n, x)$, where $\alpha$ is a standard sequence, $n$ is a standard natural number, and $x$ is a standard real number. Thus $(\alpha, n, x) \in \text{VS}$ iff $x$ is the value of $\alpha$ at $n$.

To illustrate the manner in which each relation of $^*\mathcal{R}$ is built up from the corresponding relation of $\mathcal{R}$, via the ultrafilter $U$, let us consider the less than relation $<$. We shall write "$x < y$" in place of the more cumbersome (but, from our viewpoint, more precise) "$(x, y) \in <$". Then $^*<$ is defined as

$$f \,{}^*\!\!< g \quad \text{iff} \quad \{\nu \in I \mid f(\nu) < g(\nu)\} \in U,$$

where $f$ and $g$ are maps of $I$ into $R$. For example, $1 \,{}^*\!\!< 2$ since

$$\{\nu \in I \mid 1(\nu) < 2(\nu)\} = I \in U.$$

Here 1 is a name for the map $\{(\nu, 1) \mid \nu \in I\}$ and 2 is a name for the map $\{(\nu, 2) \mid \nu \in I\}$. Notice that $f \,{}^*\!\!< 2$, where $f$ is a map of $I$ into $R$ such that $f(\nu) = 1$ for each $\nu$ in a cofinite subset of $I$.

We shall identify each standard number, indeed each member of each supporting set of $\mathcal{R}$, with the corresponding constant map of $I$ into the supporting set involved; constant maps of this sort are called *standard*. For example, we shall identify $r \in R$ with $\{(\nu, r) \mid \nu \in I\}$ and shall refer to both as *standard* real numbers. This yields an embedding of $\mathcal{R}$ into $^*\mathcal{R}$. We shall find it convenient to identify our index set $I$ with $N$, the set of all natural numbers. Under this agreement, $\omega = \{(\nu, \nu) \mid \nu \in N\}$ is a map of $I$ into $N$; thus $\omega$ is a natural number. Let us prove that $\omega$ is greater than each standard real number $r$ (i.e., $r$ is the map of $I$ into $R$ such that $r(\nu) = r$ for each $\nu \in I$). Now

$$\{\nu \in N \mid r(\nu) < \omega(\nu)\} = \{\nu \in N \mid r < \nu\};$$

this is a cofinite subset of $N$, therefore is in the ultrafilter $U$. This proves that $r < \omega$; we conclude that $\omega$ is an *infinite* natural number.

As we have observed, we can regard $^*\mathcal{R}$ as an extension of $\mathcal{R}$. We wish to prove, moreover, that $^*\mathcal{R}$ is an *elementary* extension of $\mathcal{R}$; i.e., each statement $A$ in the language of $\mathcal{R}$ is true for $\mathcal{R}$ iff $^*A$ is true for $^*\mathcal{R}$, where $^*A$ is the statement of $^*\mathcal{R}$ obtained from $A$ by starring each of its relations and interpreting each object of $\mathcal{R}$ that occurs in $A$, say $t$, as the corresponding constant map $\{(\nu, t) \mid \nu \in I\}$. In short, $^*A$ is the natural interpretation of $A$ in $^*\mathcal{R}$. If $x_1, \ldots, x_n$ are the objects of $\mathcal{R}$ that occur in a statement $A$, we write $A(x_1, \ldots, x_n)$ for $A$. Let $y_1, \ldots, y_n$ be any objects of $\mathcal{R}$. Then $A(y_1, \ldots, y_n)$ denotes the statement obtained from $A$ by replacing $x_1, \ldots, x_n$ by $y_1, \ldots, y_n$, respectively. Moreover, $^*A(f_1, \ldots, f_n)$ denotes the statement of $^*\mathcal{R}$ obtained from $A$ by starring its relations and replacing $x_1, \ldots, x_n$ by $f_1, \ldots, f_n$, respectively, provided that $f_1, \ldots, f_n$ are objects of $^*\mathcal{R}$.

To achieve our goal we need the following fact.

4.1. ŁOŚ' LEMMA. *Let $A = A(x_1, \ldots, x_n)$ be any statement of $\mathcal{R}$, and let $f_1, \ldots, f_n$ be any objects of $^*\mathcal{R}$. Then $^*A(f_1, \ldots, f_n)$ is true for $^*\mathcal{R}$ iff*

$$\{\nu \mid A(f_1(\nu), \ldots, f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U.$$

To make the logical structure of this section more evident, and to avoid drowning the reader in a flood of details, we shall postpone our proof of Łoś' Lemma to Section 5; this will also allow us to elaborate on some of the underlying ideas of the proof.

It is an immediate corollary of Łoś' Lemma that the Transfer Theorem 1.2 applies to the ultrapower $\mathcal{R}^I/U = {}^*\mathcal{R}$. This result is known as the *Ultrapower Theorem*.

4.2. THE ULTRAPOWER THEOREM. *Let $A$ be any statement in the language of $\mathcal{R}$. Then $^*A$ is true for $^*\mathcal{R}$ iff $A$ is true for $\mathcal{R}$.*

*Proof.* First we shall prove that $^*A$ is true for $^*\mathcal{R}$ in case $A$ is true for $\mathcal{R}$. Let

$$A = A(x_1, ..., x_n),$$

and let

$$g_i = \{(\nu, x_i) \mid \nu \in I\}$$

for $i = 1, ..., n$. Then $A(g_1(\nu), ..., g_n(\nu)) = A$ for each $\nu \in I$. Now

$$\{\nu \mid A(g_1(\nu), ..., g_n(\nu)) \text{ is true for } \mathcal{R}\} = \{\nu \mid A \text{ is true for } \mathcal{R}\} = I.$$

But $I \in U$; so, by Łoś's Lemma, $^*A$ is true for $^*\mathcal{R}$.

Next let $B$ be any statement in the language of $\mathcal{R}$; by applying the first part of this proof to the statement $\neg B$, we see that $B$ is true for $\mathcal{R}$ if $^*B$ is true for $^*\mathcal{R}$. This completes our proof. □

## 5. Proof of Łoś's Lemma

First we observe that Łoś's Lemma asserts that each statement $A = A(x_1, ..., x_n)$ in the language of $\mathcal{R}$ possesses a certain property; namely that for any objects $f_1, ..., f_n$ of $^*\mathcal{R}$, $^*A(f_1, ..., f_n)$ is true for $^*\mathcal{R}$ iff

$$\{\nu \mid A(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U.$$

Our point is that Łoś's Lemma declares that each statement in the language of $\mathcal{R}$ possesses a common property; for the moment, we are not concerned with the specific property involved.

The underlying idea of our proof is that if Łoś's Lemma is false, then there is a *shortest* statement of $\mathcal{R}$, say $A$, that does not have the required property. The *length* of a statement is the number of instances of logical connectives in the statement. Since the connectives mentioned in Section 2 can be expressed in terms of three connectives, $\neg$, $\vee$ and $\forall$, we can simplify our work by expressing each connective that occurs in a statement in terms of $\neg$, $\vee$ and $\forall$; then we count the instances of these connectives in the resulting statement and so obtain the length of the given statement. Of course, each atomic statement has length zero.

First we shall prove that each atomic statement has our property. It follows that our statement $A$ involves a connective, which is one of $\neg$, $\vee$ and $\forall$. Accordingly there are just three possibilities remaining: $A = \neg B$, $A = C \vee D$, $A = \forall x E$, where $B, C, D$ and $E(t)$ are statements that are each shorter than $A$ (so have the property), where $t$ is any object of $\mathcal{R}$. The idea of our proof is to demonstrate that $A$ has the property in each of these cases. It follows that each statement of $\mathcal{R}$ has the property; in other words, Łoś's Lemma is correct.

Accordingly, there are four parts to our proof. Throughout, $A$ represents a shortest statement that does not have the property of the lemma, and $f_1, ..., f_n$ are any objects of $^*\mathfrak{R}$.

*Part* 1. Assume that $A$ is atomic. Then $A$ has the form

$$(x_1, ..., x_n) \in T\,,$$

where $T$ is a relation of $\mathfrak{R}$ and $x_1, ..., x_n$ are objects of $\mathfrak{R}$. Here $^*A(f_1, ..., f_n)$ is $(f_1, ..., f_n) \in {}^*T$. By our definition of $^*T$,

$$(f_1, ..., f_n) \in {}^*T \text{ is true for } {}^*\mathfrak{R} \quad \text{iff}$$
$$\{\nu \mid (f_1(\nu), ..., f_n(\nu)) \in T \text{ is true for } \mathfrak{R}\} \in U,$$

thus

$$^*A(f_1, ..., f_n) \text{ is true for } {}^*\mathfrak{R} \quad \text{iff}$$
$$\{\nu \mid A(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathfrak{R}\} \in U.$$

We conclude that $A$ has our property.

*Part* 2. Assume that $A = \neg B$. Then $B$ has the property of the lemma, i.e.,

$$^*B(f_1, ..., f_n) \text{ is true for } {}^*\mathfrak{R} \quad \text{iff}$$
$$\{\nu \mid B(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathfrak{R}\} \in U.$$

We shall show that $A = \neg B$ has this property also. Assume that $^*A(f_1, ..., f_n)$ is true for $^*\mathfrak{R}$; but

$$^*A(f_1, ..., f_n) = \neg {}^*B(f_1, ..., f_n),$$

so $^*B(f_1, ..., f_n)$ is false for $^*\mathfrak{R}$. Therefore

$$\{\nu \mid B(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathfrak{R}\} \notin U,$$

so

$$\{\nu \mid \neg B(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathfrak{R}\} \in U$$

since $U$ is an ultrafilter. Thus

$$\{\nu \mid A(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathfrak{R}\} \in U.$$

Reading "up" this argument establishes the converse. We conclude that $\neg B$ has the property.

*Part* 3. Assume that $A = C \vee D$. Then both $C$ and $D$ have the property. Assume that $^*A(f_1, ..., f_n)$ is true for $^*\mathfrak{R}$, i.e.,

$$^*C(f_1, ..., f_n) \vee {}^*D(f_1, ..., f_n)$$

is true for $^*\mathfrak{R}$. Then one of the disjuncts, say $^*C(f_1, ..., f_n)$, is true for $^*\mathfrak{R}$. But

$C$ has our property: so

$$\{\nu \mid C(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U.$$

Thus, by the superset requirement ((1) of Section 3)

$$\{\nu \mid C(f_1(\nu), ..., f_n(\nu)) \vee D(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U. \tag{5.1}$$

To establish the converse, assume (5.1). We wish to show that

$$^*C(f_1, ..., f_n) \vee {}^*D(f_1, ..., f_n)$$

is true for $^*\mathcal{R}$. If not, then

$$\neg{}^*C(f_1, ..., f_n) \wedge \neg{}^*D(f_1, ..., f_n)$$

is true for $^*\mathcal{R}$. Since both $C$ and $D$ have our property, it follows that

$$\{\nu \mid \neg C(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U,$$

and

$$\{\nu \mid \neg D(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U.$$

Thus, by the intersection requirement,

$$\{\nu \mid \neg C(f_1(\nu), ..., f_n(\nu)) \wedge \neg D(f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U.$$

This contradicts (5.1) since $U$ is an ultrafilter. We conclude that

$$^*C(f_1, ..., f_n) \vee {}^*D(f_1, ..., f_n)$$

is true for $^*\mathcal{R}$. Thus $A$ has our property.

*Part* 4. Assume that $A = \forall x\, E(x, x_1, ..., x_n)$. Then $E(t, x_1, ..., x_n)$ has our property for each object $t$ of $\mathcal{R}$. First assume that $\forall x\, {}^*E(x, f_1, ..., f_n)$ is true for $^*\mathcal{R}$. If

$$\{\nu \mid \forall x\, E(x, f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \notin U,$$

then

$$D = \{\nu \mid \exists x\, [\neg E(x, f_1(\nu), ..., f_n(\nu))] \text{ is true for } \mathcal{R}\} \in U.$$

For each $\nu \in D$, let $a_\nu$ be an object of $\mathcal{R}$ such that putting $a_\nu$ for $x$ throughout $\neg E(x, f_1(\nu), ..., f_n(\nu))$ yields a statement true for $\mathcal{R}$. Notice that the $a_\nu$'s are not unique; accordingly, we need the Axiom of Choice to establish the existence of a function, say $h$, such that $h(\nu) = a_\nu$ for each $\nu \in D$. Clearly

$$\{\nu \mid \neg E(h(\nu), f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \in U$$

since this is a superset of $D$. By our induction assumption, $\neg{}^*E(h, f_1, ..., f_n)$ is true for $^*\mathcal{R}$; so $\exists x\, [\neg{}^*E(x, f_1, ..., f_n)]$ is true for $^*\mathcal{R}$, i.e., $\forall x\, {}^*E(x, f_1, ..., f_n)$ is false for $^*\mathcal{R}$. This contradiction proves that

(5.2) $\{\nu \mid \forall x\, E(x, f_1(\nu), ..., f_n(\nu))$ is true for $\mathcal{R}\} \in U$.

Next assume (5.2). If $\forall x\, [^*E(x, f_1, ..., f_n)]$ is not true for $^*\mathcal{R}$, then $\exists x\, [\neg {}^*E(x, f_1, ..., f_n)]$ is true for $^*\mathcal{R}$. Thus there is an object $g$ of $^*\mathcal{R}$ such that $\neg {}^*E(g, f_1, ..., f_n)$ is true for $^*\mathcal{R}$. But $E = E(t, x_1, ..., x_n)$ has our property; so

$$\{\nu \mid E(g(\nu), f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \notin U.$$

Notice that $E(g(\nu), f_1(\nu), ..., f_n(\nu))$ is true for $\mathcal{R}$ if $\forall x\, E(x, f_1(\nu), ..., f_n(\nu))$ is true for $\mathcal{R}$; thus

$$\{\nu \mid \forall x\, E(x, f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \subset \{\nu \mid E(g(\nu), f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\}..$$

Therefore

$$\{\nu \mid \forall x\, E(x, f_1(\nu), ..., f_n(\nu)) \text{ is true for } \mathcal{R}\} \notin U.$$

This contradiction proves that $\forall x\, [^*E(x, f_1, ..., f_n)]$ is true for $^*\mathcal{R}$. Thus $A$ has our property.

Since we have exhausted all possibilities, we conclude that each statement in the language of $\mathcal{R}$ has our property. This completes our proof of Łoś's Lemma. □

## 6. $^*\mathcal{R}$ is sequentially comprehensive

The purpose of this section is to prove the following fact about the ultrapower $^*\mathcal{R}$, namely that each sequence, over $N$, of objects of $^*\mathcal{R}$ can be extended to an internal sequence of $^*N$ (see Robinson [1973]).

6.1. LEMMA. *Let $B$ be a supporting set of $^*\mathcal{R}$ and let $g_n \in B$ for each $n \in N$. In $^*\mathcal{R}$ there is an internal sequence $(s_n)_{n \in {}^*N}$, such that $s_n = g_n$ for each $n \in N$.*

*Proof.* For simplicity we treat the case in which $B$ is $^*R$; our argument, however, easily extends to the case of sequences over any supporting set of $^*\mathcal{R}$. So we shall regard the given sequence $(g_n)$ as a sequence of real numbers, i.e., members of $^*\mathcal{R}$, where $n \in N$. In view of the ultrapower construction (see Section 4) we may interpret a real number as a map of $I$ into $R$; thus $(g_n)$ is a sequence of maps.

As we have mentioned earlier, the set of all sequences over $R$, which we denote by Seq, is a supporting set of $\mathcal{R}$. The corresponding supporting set $^*$Seq of $^*\mathcal{R}$ is the set of all internal sequences over $^*R$ (this is not the set of all sequences over $^*R$; recall the distinction between internal and external). In Section 4 we introduced VS, the relation of $\mathcal{R}$ that expresses the value of a sequence at a standard natural number; $^*$VS is the corresponding relation of $^*\mathcal{R}$. Now a sequence in $^*\mathcal{R}$ is a map of $I$ into Seq, the set of all sequences whose terms are standard numbers. Let $s$ be the sequence such that $s(\nu) = (g_n(\nu))$ for each $\nu \in I$; i.e.,

$$s(\nu) = (g_1(\nu), g_2(\nu), g_3(\nu), ...),$$

a member of Seq. We claim that $s$ is an internal sequence of $^*\mathcal{R}$ and that $s_n$, its value at $n$, is $g_n$ for each $n \in N$.

First we shall prove that $s$ is internal. Let $n$ be any map of $I$ into $N$ (i.e., $n$ is a natural number), and let $v$ be the map of $I$ into $R$ such that $v(\nu) = g_{n(\nu)}(\nu)$ for each $\nu \in I$. Of course, $(s(\nu), n(\nu), v(\nu)) \in \text{VS}$ for each $\nu \in I$; so

$$\{\nu \mid (s(\nu), n(\nu), v(\nu)) \in \text{VS}\} = I \in U.$$

We conclude that $(s, n, v) \in {}^*\text{VS}$.

We point out that the first term of each member of $^*$VS is an internal sequence. This is due to two facts:

(i) the first term of each member of VS is a sequence;

(ii) (i) can be expressed in the language of $\mathcal{R}$.

Let us make sure of this point. Consider the following statement in the language of $\mathcal{R}$:

$$(6.2) \qquad \forall x\, [\exists yz\, [(x, y, z) \in \text{VS}] \to x \in \text{Seq}],$$

where the quantifiers refer to the basic set of $\mathcal{R}$. To see that (6.2) is true for $\mathcal{R}$, it is sufficient to appeal to the definition of the relation VS. By construction, a triple is a member of VS iff its first term is a standard sequence, its second term is a standard natural number, and its third term is the value of the sequence at that natural number. So the set of first terms of VS is the set of standard sequences, namely Seq. Thus (6.2) is true for $\mathcal{R}$. Interpreting (6.2) in $^*\mathcal{R}$ yields

$$(6.3) \qquad \forall x\, [\exists yz\, [(x, y, z) \in {}^*\text{VS}] \to x \in {}^*\text{Seq}],$$

which is true for $^*\mathcal{R}$ by the Ultrapower Theorem 4.2. Therefore the first term of each member of $^*$VS is an internal sequence. Since $(s, n, v) \in {}^*\text{VS}$, we conclude that $s \in {}^*\text{Seq}$; i.e., $s$ is an internal sequence of $^*\mathcal{R}$.

Next we shall show that the value of $s$ at $m$ is $g_m$ for each $m \in N$. Here the first $m$ is the constant map that associates the standard natural number $m$ with each member of $I$; i.e., $m(\nu) = m$ for each $\nu \in I$. Define $v$ as above, i.e., $v(\nu) = g_{m(\nu)}(\nu)$ for each $\nu \in I$. Then

$$(s(\nu), m(\nu), v(\nu)) = (s(\nu), m, g_m(\nu)).$$

Clearly, the value of the sequence $(g_1(\nu), g_2(\nu), g_3(\nu), \ldots)$ at $m$ is $g_m(\nu)$. So, for each $\nu \in I$, $(s(\nu), m(\nu), v(\nu)) \in \mathrm{VS}$; i.e.,

$$\{\nu \mid (s(\nu), m(\nu), v(\nu)) \in \mathrm{VS}\} = I \in U.$$

Thus $(s, m, v) \in {}^*\mathrm{VS}$; this means that $v$ is the value of $s$ at $m$ (in symbols, $s_m = v$). But for each $\nu \in I$,

$$v(\nu) = g_{m(\nu)}(\nu) = g_m(\nu),$$

so $v = g_m$. This proves that $s_m = g_m$ for each $m \in N$, and completes the proof of our lemma. □

The lemma asserts that each sequence $(g_n)$ over $N$, of objects of ${}^*\mathcal{R}$ from the same supporting set, can be extended to an internal sequence of ${}^*\mathcal{R}$. We describe this property of ${}^*\mathcal{R}$ by saying that ${}^*\mathcal{R}$ is *sequentially comprehensive*. It is important to know that a sequence over $N$ can be extended to an internal sequence over ${}^*N$ because of the central importance of internal entities. Moreover, each internal sequence yields many internal subsets of ${}^*N$. For example, if $s$ is an internal sequence, then both $\{n \mid n \in {}^*N \wedge s_n = 1\}$ and $\{n \mid n \in {}^*N \wedge 0 < s_n < 1\}$ are internal subsets of ${}^*N$.

In this regard we bring out another method of generating internal sequences of ${}^*\mathcal{R}$. Notice that in $\mathcal{R}$, corresponding to each $t \in R$ there is a sequence $(a_n)$ defined as follows:

$$a_n = \begin{cases} 0 & \text{if } n < t, \\ 1 & \text{if } n \geqslant t. \end{cases}$$

This fact can be expressed in the language of $\mathcal{R}$ as follows:

(6.4) $\forall t x\, [\forall n\, [n < t \rightarrow (x,n,0) \in \mathrm{VS}] \wedge \forall n\, [n \geqslant t \rightarrow (x,n,1) \in \mathrm{VS}] \rightarrow x \in \mathrm{Seq}]$.

Here the first quantifier refers to $R$, the second to the basic set of $\mathcal{R}$, and the remaining quantifiers to $N$.

Since (6.4) is true for $\mathcal{R}$, its interpretation in ${}^*\mathcal{R}$ is true for ${}^*\mathcal{R}$. So the following statement is true for ${}^*\mathcal{R}$:

(6.5) $\forall t x\, [\forall n\, [n < t \rightarrow (x,n,0) \in {}^*\mathrm{VS}] \wedge \forall n\, [n \geqslant t \rightarrow (x,n,1) \in {}^*\mathrm{VS}] \rightarrow x \in {}^*\mathrm{Seq}]$.

Here the first quantifier refers to ${}^*R$, the second to the basic set of ${}^*\mathcal{R}$, and the remaining quantifiers to ${}^*N$.

Since (6.5) is true for ${}^*\mathcal{R}$, so is each statement obtained from (6.5) by deleting "$\forall t$" and replacing the remaining $t$'s by a specific member of ${}^*R$. In particular, we obtain that $(s_n)$ is an internal sequence of $^*\mathcal{R}$, where

$$s_n = \begin{cases} 0 & \text{if } n < \omega, \\ 1 & \text{if } n \geqslant \omega, \end{cases}$$

here $\omega$ is the infinite natural number defined in Section 4. Thus $(s_n)$ is an internal sequence of $^*\mathcal{R}$.

On the other hand, the map $(t_n)$ for which

$$t_n = \begin{cases} 0 & \text{if } n \in N, \\ 1 & \text{if } n \in {}^*N - N \end{cases}$$

is an external sequence of $^*\mathcal{R}$, i.e., $(t_n) \notin {}^*\text{Seq}$.

## 7. Principles of permanence

In the remainder of this book, by $^*\mathcal{R}$ we shall mean any sequentially comprehensive model of the postulate set of Section 1, not necessarily the ultrapower $^*\mathcal{R}$ of Section 4.

We have agreed to call each member of $^*(\mathcal{P}N)$ an *internal* subset of $^*N$; and we have agreed to call any other member of $\mathcal{P}(^*N)$ an *external* subset of $^*N$. The Transfer Theorem 1.2 allows us to establish certain facts about $^*(\mathcal{P}N)$ by merely observing that $\mathcal{P}N$ has the corresponding property in $\mathcal{R}$ (i.e., the corresponding statement about $\mathcal{P}N$, in the language of $\mathcal{R}$, is true for $\mathcal{R}$).

To illustrate this fact, we mention that Peano's induction postulate (see (1.8)) yields the following statement.

7.1. PRINCIPLE OF MATHEMATICAL INDUCTION FOR $^*\mathcal{R}$. *If $S$ is an internal subset of $^*N$ such that $1 \in S$ and $\forall x\, [x \in S \to x + 1 \in S]$, then $S = {}^*N$.*

*Proof.* In effect, we are quantifying over $^*(\mathcal{P}N)$, which we regard as a supporting set of $^*\mathcal{R}$. The corresponding statement in the language of $\mathcal{R}$ is true for $\mathcal{R}$; so, by the Transfer Theorem 1.2, our statement is true for $^*\mathcal{R}$. □

By a *principle of permanence* we mean a statement which declares that if each member of a set $B$ has a stated property, then each member of some superset of $B$ has the property.

7.2. FIRST PRINCIPLE OF PERMANENCE. *Let $A$ be an internal subset of $^*N$ that includes $N$. Then there is an infinite natural number $\kappa$ such that $n \in A$ for each $n \leqslant \kappa$, $n \in {}^*N$.*

*Proof.* In $\mathcal{R}$, each nonempty subset of $N$ has a smallest member; therefore, in $^*\mathcal{R}$, each nonempty internal subset of $^*N$ has a smallest member. Now either $A = {}^*N$ or $A$ is a proper subset of $^*N$. In the former case, there is nothing to prove. In the latter case, $^*N - A$ is a nonempty internal subset of $^*N$. Therefore, by the observation that begins this proof, $^*N - A$ has a smallest member, which must be infinite, say $\kappa + 1$. Thus $n \in A$ for each $n \leqslant \kappa$, $n \in {}^*N$. This establishes our principle of permanence. □

7.3. SECOND PRINCIPLE OF PERMANENCE. *Let $A$ be an internal subset of $^*N$ that contains each infinite natural number. Then there is a finite natural number $q$ such that $n \in A$ for each $n > q$, $n \in {}^*N$.*

*Proof.* In $\mathcal{R}$, each nonempty subset of $N$ that is bounded above has a largest member; therefore, in $^*\mathcal{R}$, each nonempty internal subset of $^*N$ that is bounded above, has a largest member. Now, either $A = {}^*N$ or there is a finite natural number that is not in $A$. In the former case, there is nothing to prove. In the latter case, $^*N - A$ is a nonempty internal subset of $^*N$ which is bounded above (by each infinite natural number). So, by the observation that begins this proof, $^*N - A$ has a largest member, say $q$. We conclude that $n \in A$ for each $n > q$, $n \in {}^*N$. This establishes our principle of permanence. □

For our next principle of permanence we shall require the following lemma, which expresses a fact about internal sequences.

7.4. LEMMA. *Let $(s_n)$ be an internal sequence of $^*\mathcal{R}$, and let $m \in {}^*R$ be such that $|s_n| \leqslant m$ for each $n \in N$. Then there is an infinite natural number $\kappa$ such that $|s_n| \leqslant m$ for each $n < \kappa$.*

*Proof.* If $|s_n| \leqslant m$ for each $n \in {}^*N$, there is nothing to prove. Otherwise, let

$$A = \{n \mid n \in {}^*N \wedge |s_n| > m\}.$$

Certainly, $A$ is a nonempty internal subset of $^*N$; thus $A$ has a smallest member, say $\kappa$, which of course is infinite. We conclude that $|s_n| \leqslant m$ for each $n < \kappa$, $n \in {}^*N$. □

We are now ready for the

7.5. THIRD PRINCIPLE OF PERMANENCE. *Let $(s_n)$ be an internal sequence of $^*\mathcal{R}$ such that $s_n \simeq 0$ for each $n \in N$. Then there is an infinite natural number $\kappa$ such that $s_n \simeq 0$ for each $n < \kappa$, $n \in {}^*N$.*

*Proof.* We point out that $(n\, s_n)$ is an internal sequence of ${}^*\mathcal{R}$. Let $n \in N$; then $n\, s_n \simeq 0$, so $|n\, s_n| \leqslant 1$. Thus, by Lemma 7.4, there is an infinite natural number $\kappa$ such that $|s_n| \leqslant 1/n$ for each $n < \kappa$, $n \in {}^*N$. Thus $s_n \simeq 0$ if $n < \kappa$ and $n$ is infinite. By assumption, $s_n \simeq 0$ for each $n \in N$. This completes our proof. □

7.6. COROLLARY. *Let $(s_n)$ be an internal sequence of ${}^*\mathcal{R}$ such that $s_n$ is infinite if $n \in N$. Then there is an infinite natural number $\kappa$ such that $s_n$ is infinite for each $n < \kappa$, $n \in {}^*N$.*

*Proof.* The sequence $(1/s_n)$ is internal, and $1/s_n \simeq 0$ for each $n \in N$ (clearly, $1/s_n \neq 0$ if $n \in N$). So, by the Third Principle of Permanence, there is an infinite natural number $\kappa$ such that $1/s_n \simeq 0$ for each $n < \kappa$, $n \in {}^*N$. The reciprocal of a nonzero infinitesimal is infinite; we conclude that $s_n$ is infinite if $n < \kappa$, $n \in {}^*N$. □

We can also use the Third Principle of Permanence to establish a result of a different character (see Robinson [1973], Theorem 4.1).

7.7. LEMMA. *Let $(a_n)$ be a decreasing sequence, over N, of infinite natural numbers; i.e., $a_n \geqslant a_{n+1}$ and $a_n \in {}^*N - N$ for each $n \in N$. Then there is an infinite natural number $\kappa$ such that $a_n > \kappa$ for each $n \in N$.*

*Proof.* Since ${}^*\mathcal{R}$ is sequentially comprehensive, there is an internal sequence $(s_n)$ such that $s_n = a_n$ for each $n \in N$. Let $(u_n)$ be the internal sequence obtained from $(s_n)$ by replacing each of its zero terms (if any) by 1. Next let $(t_n)$ be the internal sequence such that for each $n \in {}^*N$,

$$t_n = n/\min\{|u_1|, ..., |u_n|\}.$$

For each $n \in N$, $t_n = n/a_n$; so $t_n \simeq 0$. By the Third Principle of Permanence there is an infinite natural number, say $\kappa + 1$, such that $t_n \simeq 0$ for each $n < \kappa + 1$, $n \in {}^*N$. In particular, $t_\kappa \simeq 0$, i.e.,

$$\kappa/\min\{|u_1|, ..., |u_\kappa|\} \simeq 0,$$

so $|u_n| > \kappa$ for each $n \leqslant \kappa$, $n \in {}^*N$; thus $a_n > \kappa$ for each $n \in N$. □

Returning to our proof of Lemma 7.7, notice that $\kappa/a_n \simeq 0$ for each $n \in N$. This points out the fact that ${}^*N - N$ is so extensive that each of its members, indeed any sequence of its members, is preceded by an infinite natural number so small that its ratio to each member of the sequence is infinitesimal.

7.8. COROLLARY. *Let $(a_n)$ be a decreasing sequence, over $N$, of infinite real numbers. Then there is an infinite natural number $\kappa$ such that $a_n > \kappa$ for each $n \in N$.*

*Proof.* For each $n \in N$, replace $a_n$ by the largest natural number that does not exceed $a_n$. Apply Lemma 7.7 to the resulting sequence. □

Now we can prove the following fact.

7.9. LEMMA. *Let $(a_n)$ be an increasing sequence, over $N$, of positive infinitesimals. Then there is an infinitesimal $\epsilon$ such that $a_n < \epsilon$ for each $n \in N$.*

*Proof.* By assumption, $(1/a_n)$ is a decreasing sequence, over $N$, of infinite real numbers. By Corollary 7.8 there is an infinite natural number $\kappa$ such that $1/a_n > \kappa$ for each $n \in N$. Let $\epsilon = 1/\kappa$; then $\epsilon \simeq 0$ and $a_n < \epsilon$ for each $n \in N$. □

Here is an example.

7.10. EXAMPLE. Let $\rho$ be any positive infinitesimal. Then $(\rho^{1/n})$, where $n \in N$, is an increasing sequence, over $N$, of positive infinitesimals; so, by Lemma 7.9, there is an infinitesimal $\epsilon$ such that $\epsilon > \rho^{1/n}$ for each $n \in N$. We conclude that corresponding to each positive infinitesimal $\rho$ there is an infinitesimal $\epsilon$ so large compared with $\rho$ that $\epsilon^n > \rho$ for each $n \in N$.

Our next lemma should be compared to Corollary 7.8.

7.11. LEMMA. *Let $(a_n)$ be an increasing sequence, over $N$, of infinite real numbers. Then there is an infinite natural number $\kappa$ such that $a_n < \kappa$ for each $n \in N$.*

*Proof.* Since ${}^*\mathcal{R}$ is sequentially comprehensive, there is an internal sequence $(s_n)$ such that $s_n = a_n$ for each $n \in N$. For each $n \in {}^*N$, let $t_n = \max\{s_1, ..., s_n\}$; clearly $(t_n)$ is an internal sequence. Now $\omega$ is an infinite natural number; so $t_\omega \geqslant t_n$ for each $n < \omega$. Certainly there is a natural number greater than $t_\omega$, say $\kappa$. Then $t_n < \kappa$ for each $n < \omega$; thus $a_n < \kappa$ for each $n \in N$. □

This result has the following corollary.

7.12. COROLLARY. *Let $(a_n)$ be a decreasing sequence, over $N$, of positive infinitesimals. Then there is a positive infinitesimal $\epsilon$ such that $a_n > \epsilon$ for each $n \in N$.*

*Proof.* Apply Lemma 7.11 to the sequence of reciprocals $(1/a_n)$. □

Our principles of permanence can be formulated in terms of the notion of a *property*. Now, a property of natural numbers is characterized by the set that consists of all natural numbers which possess the property.

Here are some examples, replacing sets by properties.

7.13. EXAMPLE. Let $(s_n)$ be an internal sequence of $^*\mathcal{R}$ such that $|s_n| > m$ for each $n \in N$. We shall say that a natural number $t$ has property $P$ if $|s_t| > m$. This is an internal property since the corresponding subset of $^*N$ is internal. Therefore, by the First Principle of Permanence, there is an infinite natural number $\kappa$ such that each natural number less than $\kappa$ has the property; i.e., $|s_n| > m$ if $n < \kappa$, $n \in {}^*N$.

7.14. EXAMPLE. Let $(s_n)$ be an internal sequence of $^*\mathcal{R}$ such that $s_n = 1$ for each $n \in {}^*N - N$. We say that a natural number $t$ has property $P$ if $s_t = 1$; this is an internal property. Therefore, by the Second Principle of Permanence, there is a finite natural number $q$ such that each natural number greater than $q$ has the property; i.e., $s_n = 1$ for each $n > q$, $n \in {}^*N$.

We present one more lemma.

7.15. LEMMA. *Let $(s_n)$ be an internal sequence of $^*\mathcal{R}$ such that $|s_n| \leqslant m$ for each $n \in {}^*N - N$, where $m \in {}^*R$. Then there is a finite natural number $q$ such that $|s_n| \leqslant m$ for each $n > q$.*

*Proof.* If $|s_n| \leqslant m$ for each $n \in {}^*N$, there is nothing to prove. Otherwise, let

$$A = \{n \mid n \in {}^*N \wedge |s_n| > m\}.$$

Now $A$ is a nonempty internal subset of $^*N$; moreover, $A$ is bounded above by each infinite natural number. It follows that $A$ has a largest member, say $q$, which of necessity is finite. We conclude that $|s_n| \leqslant m$ for each $n > q$. □

Finally, we mention that principles of permanence must be formulated with some care. The following statement, which has the form of a principle of permanence, is clearly false. "Let $(s_n)$ be an internal sequence such that $s_n$ is finite for each $n \in N$. Then there is an infinite natural number $\kappa$ such that $s_n$ is finite for each $n < \kappa$." To see that this statement is false, apply it to the internal sequence $(n)$; i.e., take $s_n = n$ for each $n \in {}^*N$.

Here is another false principle of permanence. "Let $(s_n)$ be an internal sequence such that $s_n \simeq 0$ for each infinite $n$. Then there is a finite natural number $q$ such that $s_n \simeq 0$ for each $n > q$, $n \in {}^*N$." To see that this statement is false, take $(s_n) = (1/n)$.

## 8. Continuity in $\mathcal{R}$

The purpose of this section is to prepare the way for the discussion of continuity in ${}^*\mathcal{R}$ presented in Section 10.

The notion that a function $f$ is continuous at a standard number $a$ in its domain was characterized by Weierstrass as follows:

$$(8.1) \qquad \forall\epsilon\,\exists\delta\,\forall x\,[\,|x-a|<\delta \to |f(x)-f(a)|<\epsilon\,].$$

Here the Greek letters indicate quantification over positive standard numbers, and the last universal quantifier refers to the domain of $f$.

By utilizing certain concepts of ${}^*\mathcal{R}$, we can characterize continuity in terms that follow our intuition; i.e., we say that $f$ is continuous at $a$, where $f \in F$ and $a \in \operatorname{dom} f$, provided that ${}^*f(x)$ is infinitely close to ${}^*f(a)$ whenever $x$ is infinitely close to $a$. In symbols,

$$(8.2) \qquad \forall x\,[x \simeq a \to {}^*f(x) \simeq {}^*f(a)],$$

where the quantifier refers to the domain of ${}^*f$ in ${}^*\mathcal{R}$.

Notice that (8.2) involves $\simeq$, a relation of ${}^*\mathcal{R}$ that corresponds to no relation of $\mathcal{R}$. So, removing stars throughout (8.2) does not yield a statement in the language of $\mathcal{R}$; i.e., (8.2) corresponds to no statement of $\mathcal{R}$. This observation explains the difficulty in characterizing continuity when ${}^*\mathcal{R}$ is not available. It took the genius and insight of many mathematicians (e.g., Eudoxus, Euclid, Archimedes, Bolzano, Weierstrass) before it was appreciated that (8.1) characterizes this notion.

We shall now show that (8.1) and (8.2) are mathematically equivalent.

8.3. LEMMA. *Let $f \in F$, and let $a \in$* dom$f$. *Then* (8.1) *is true for* $\mathcal{R}$ *iff* (8.2) *is true for* ${}^*\mathcal{R}$.

*Proof.* (i) Assume that (8.1) is true for $\mathcal{R}$. Let $h$ be any positive standard number; by (8.1) there is a positive standard number $t$ such that

$$(8.4) \qquad \forall x\,[\,|x-a|<t \to |f(x)-f(a)|<h\,]$$

is true for $\mathcal{R}$. Therefore (8.4) is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$. Now consider (8.2). Let $x \simeq a$, and let $x$ be in the domain of ${}^*f$; then $|x-a|$ is less than each positive standard number, so $|x-a|<t$. Thus, by (8.4) interpreted in ${}^*\mathcal{R}$,

$$|f(x)-f(a)|<h.$$

Since $h$ is any positive standard number, we conclude that $f(x)-f(a)$ is an infinitesimal, so $f(x)\simeq f(a)$. Therefore (8.2) is true for ${}^*\mathcal{R}$.

(ii) Assume that (8.2) is true for ${}^*\mathcal{R}$. Let $h$ be any positive standard number. We claim that

$$\exists\delta\,\forall x\,[|x-a|<\delta\rightarrow|f(x)-f(a)|<h] \tag{8.5}$$

is true for ${}^*\mathcal{R}$. To see this, take any infinitesimal for $\delta$. If $|x-a|<\delta$, then $x\simeq a$, so by (8.2),

$$f(x)\simeq f(a),$$

and it follows that

$$|f(x)-f(a)|<h.$$

By Transfer Theorem 1.2, (8.5) is true for $\mathcal{R}$ when interpreted in $\mathcal{R}$; i.e.,

$$\exists\delta\,\forall x\,[|x-a|<\delta\rightarrow|f(x)-f(a)|<h] \tag{8.6}$$

is true for $\mathcal{R}$. Here $h$ is any positive standard number; i.e., (8.6) is true for $\mathcal{R}$ for each positive standard number $h$. Thus

$$\forall\epsilon\,\exists\delta\,\forall x\,[|x-a|<\delta\rightarrow|f(x)-f(a)|<\epsilon] \tag{8.7}$$

is true for $\mathcal{R}$. This completes our proof of the lemma. □

Weierstrass characterized continuity of $f$ (as opposed to continuity of $f$ at a member of its domain) as follows: $f$ is *continuous* iff

$$\forall y\epsilon\,\exists\delta\,\forall x\,[x-y|<\delta\rightarrow|f(x)-f(y)|<\epsilon] \tag{8.8}$$

is true for $\mathcal{R}$. Again, the Greek letters indicate quantification over positive standard numbers, and the Latin letters indicate quantification over the domain of $f$.

Quantifying the "$a$" of (8.2) yields the corresponding nonstandard formulation; i.e., $f$ is continuous, where $f\in F$, iff

$$\forall ax\,[x\simeq a\rightarrow f(x)\simeq f(a)] \tag{8.9}$$

is true for ${}^*\mathcal{R}$. Here the first quantifier refers to standard numbers in the domain of $f$, and the second quantifier refers to the domain of $f$.

The value of the nonstandard approach is particularly evident in the case of uniform continuity. The Weierstrass characterization is as follows: a function $f$ is *uniformly continuous* iff

$$(8.10) \qquad \forall\epsilon\, \exists\delta\, \forall xy\, [\,|x-y| < \delta \to |f(x) - f(y)| < \epsilon]$$

is true for $\mathcal{R}$, under the same conventions on quantifiers as for (8.8).

Comparing (8.8) and (8.10), we see that for continuity Weierstrass requires that corresponding to each $y$ there is a suitable $\delta$, whereas for uniform continuity Weierstrass requires much more, namely that a $\delta$ exists which is suitable for each $y$. In short, the condition for uniform continuity can be obtained from the condition for continuity by moving the quantifier $\forall y$ to the right of $\exists\delta$.

Using the concepts of $^*\mathcal{R}$ we formulate uniform continuity as follows: a function $f \in F$ is *uniformly continuous* iff

$$(8.11) \qquad \forall xy\, [x \simeq y \to f(x) \simeq f(y)]$$

is true for $^*\mathcal{R}$; here the quantifiers refer to the domain of $f$.

Contrasting (8.9) and (8.11), we see that the distinction between continuity and uniform continuity reduces to the domain of a single quantifier, the $\forall a$ of (8.9). For continuity, this quantifier refers to standard members of the domain of $f$; for uniform continuity, the corresponding quantifier of (8.11) refers to the entire domain of $f$.

The two characterizations of uniform continuity are mathematically equivalent:

8.12. LEMMA. *Let* $f \in F$. *Then* (8.10) *is true for* $\mathcal{R}$ *iff* (8.11) *is true for* $^*\mathcal{R}$.

*Proof.* Follow the pattern of the proof of Lemma 8.3. □

The power of Nonstandard Analysis is revealed by the ease and directness of our proofs of key facts concerning such notions as continuity, limits and series. To illustrate this, we shall prove the well-known fact that a continuous function is uniformly continuous if its domain is a finite closed interval; by *finite* we mean that the closed interval contains only finite numbers.

8.13. LEMMA. *Let $f$ be a continuous function whose domain is a finite closed interval $I$. Then $f$ is uniformly continuous.*

*Proof.* Let $a \simeq b$, $a, b \in {}^*I$; then $a$ and $b$ are finite. By the Fundamental Theorem about Finite Numbers 1.7 there is a standard number $c \in {}^*I$ such that

$c \simeq a$ and $c \simeq b$. But $f$ is continuous at $c$, so

$$f(c) \simeq f(a), \qquad f(c) \simeq f(b).$$

Therefore $f(a) \simeq f(b)$. In view of (8.11) this proves that $f$ is uniformly continuous. □

Next we shall use the resources of $^*\mathcal{R}$ to characterize the *limit* of a sequence and the *sum* of a series.

8.14. CRITERION FOR THE LIMIT OF A SEQUENCE. *Let* $(a_n) \in \text{Seq}$ *and* $L \in R$. *Then* $(a_n)$ *converges and* $\lim(a_n) = L$ *iff* $a_\kappa \simeq L$ *for each* $\kappa \in {}^*N - N$.

The proof is left to the reader.

We can simplify this criterion by dropping the requirement that each $a_\kappa$ is finite. Thus we claim that $(a_n)$ *converges* iff $a_\kappa \simeq a_\omega$ for each $\kappa \in {}^*N - N$. Moreover, if $(a_n)$ converges under this criterion, then we say that $\lim(a_n) = {}^0a_\omega$. Of course, we must prove that $a_\omega$ is finite if $(a_n)$ converges.

8.15. LEMMA. *Let* $(a_n) \in \text{Seq}$, *and let* $a_\kappa \simeq a_\omega$ *for each* $\kappa \in {}^*N - N$. *Then* $a_\omega$ *is finite.*

*Proof.* The sequence $(a_n)$ extends to an internal sequence of $^*\mathcal{R}$ which we can denote by $^*(a_n)$, or, following the usual mathematical convention, simply by $(a_n)$. Certainly, the difference of two internal sequences is an internal sequence; so $(a_n - a_\omega)$ is an internal sequence. By assumption, $|a_\kappa - a_\omega| < 1$ for each $\kappa \in {}^*N - N$. Therefore, by Lemma 7.15, there is a finite natural number $q$ such that $|a_n - a_\omega| < 1$ for each $n > q$. In particular, $|a_{q+1} - a_\omega| < 1$; here $a_{q+1}$ is finite, so $a_\omega$ is finite.

8.16. CRITERION FOR THE SUM OF A SERIES. *Let* $\Sigma_N a_n$ *be any series in* $\mathcal{R}$, *and let* $S \in R$. *Then* $\Sigma_N a_n$ *converges and* $\Sigma_N a_n = S$ *iff*

$$a_1 + \ldots + a_\kappa \simeq S$$

*for each* $\kappa \in {}^*N - N$.

Again, we can simplify our criterion by dropping the requirement that each sum $a_1 + \ldots + a_\kappa$ is finite. Thus we say that $\Sigma_N a_n$ *converges* iff

$$a_1 + \ldots + a_\kappa \simeq a_1 + \ldots + a_\omega$$

for each $\kappa \in {}^*N - N$. Moreover, if $\Sigma_N a_n$ converges under this criterion, then we say that

$$\sum_N a_n = {}^0(a_1 + \ldots + a_\omega).$$

Of course, we must prove that $a_1 + \ldots + a_\omega$ is finite if $\Sigma_N a_n$ converges.

8.17. LEMMA. *Let* $\Sigma_N a_n$ *be any series in* $\mathcal{R}$, *and let*

$$a_1 + \ldots + a_\kappa \simeq a_1 + \ldots + a_\omega$$

*for each* $\kappa \in {}^*N - N$. *Then* $a_1 + \ldots + a_\omega$ *is finite.*

*Proof.* Follow the pattern of the proof of Lemma 8.15. □

## 9. Internal functions

It is important to realize that ${}^*F$, the set of all internal functions of ${}^*\mathcal{R}$, is an extension of $F$, the set of all functions of $\mathcal{R}$, in two senses. First, certain members of $F$ extend their domains and ranges in the passage from $\mathcal{R}$ to ${}^*\mathcal{R}$. For example, the identity function $\{(t, t) \mid t \in R\}$ grows to the identity function $\{(t, t) \mid t \in {}^*R\}$ which is a member of ${}^*F$. We mention that a finite function such as $\{(1, 0), (2, 5)\}$ does not grow in the passage from $\mathcal{R}$ to ${}^*\mathcal{R}$.

Secondly, ${}^*F$ contains functions that are not rooted in $F$ in this way. Here is an example.

9.1. EXAMPLE. The mapping $\{(t, \omega) \mid t \in {}^*R\}$ is an internal function of ${}^*\mathcal{R}$; here $\omega$ is the infinite natural number introduced in Section 1. Why? Because the following statement is true for $\mathcal{R}$ and can be expressed in the language of $\mathcal{R}$:

(9.2) Corresponding to each $a \in R$ there is a function $f$ such that $f(t) = a$ for each $t \in R$.

By the Transfer Theorem, (9.2) is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$. The universal quantifiers of (9.2) refer to $R$, and its existential quantifier refers to $F$. To interpret (9.2) in ${}^*\mathcal{R}$ we must star its relations, and interpret its quantifiers appropriately, i.e., the universal quantifiers refer to ${}^*R$ and the existential quantifier refers to ${}^*F$. It follows, in particular, that $\{(t, \omega) \mid t \in {}^*R\} \in {}^*F$; i.e., $\{(t, \omega) \mid t \in {}^*R\}$ is an internal function. Clearly this function is *not* rooted in any member of $F$.

We must clarify the first paragraph of this section. The basic fact is that corresponding to *each* $f \in F$ there is an internal function, say ${}^*f$, which possesses all properties of $f$ that can be expressed in the language of $\mathcal{R}$. Moreover, and this is important, ${}^*f$ is a superset of $f$. This is due to the fact that the statement $(a, b) \in f$ is in the language of $\mathcal{R}$. Therefore, if $(a, b) \in f$ is true for $\mathcal{R}$, it is also true for ${}^*\mathcal{R}$; i.e., $(a, b) \in {}^*f$. So ${}^*f$ is a superset of $f$. Of course, this is merely a necessary condition for an internal function of the first kind and is not a sufficient condition. Frequently, though, this necessary condition allows us to demonstrate that a given internal function has not grown from a member of $F$.

We obtain more examples of the second kind of internal functions by quoting suitable methods of constructing members of $F$ from given members of $F$ and members of $R$ (it is the presence of the latter that is essential here). The following statement is true for $\mathcal{R}$ and can be expressed in the language of $\mathcal{R}$:

(9.3) Corresponding to each function $f$ with domain $R$, and to each number $a$, there is a function $g$ such that $g(t) = f(at)$ for each $t \in R$.

It follows that (9.3) is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$. The universal quantifiers of (9.3) refer to $F$ and $R$, and its existential quantifier refers to $F$. Therefore, interpreting (9.3) in ${}^*\mathcal{R}$ involves quantifying over ${}^*F$ and ${}^*R$; in particular, its existential quantifier refers to ${}^*F$.

Here are two internal functions that are obtained by applying (9.3) in ${}^*\mathcal{R}$.

9.4. EXAMPLE. The function $\sin \in F$ grows to a member of ${}^*F$ that we shall also denote by sin, following the usual mathematical convention. Therefore, by (9.3) interpreted in ${}^*\mathcal{R}$,

$$\{(t, \sin \omega t) \mid t \in {}^*R\}$$

is an internal function of ${}^*\mathcal{R}$. We shall denote this function, for later reference, by $\sin \omega x$. We mention that $\sin \omega x$ is *not* rooted in any member of $F$.

9.5. EXAMPLE. The function $\exp x$ or $e^x$ of $\mathcal{R}$ grows to a member of ${}^*F$ that we shall also denote by $\exp x$ or $e^x$. Moreover, $e^{-x^2} \in F$ and this function grows in the passage to ${}^*\mathcal{R}$. Therefore, by (9.3) interpreted in ${}^*\mathcal{R}$ (with $a = \sqrt{\omega}$),

$$\{(t, \exp[-\omega t^2]) \mid t \in {}^*R\} \in {}^*F;$$

i.e., $\exp[-\omega x^2]$ is an internal function. Since $\exp[-\omega t^2]$ is an infinitesimal for each non-zero $t \in R$, we conclude that $\exp[-\omega x^2]$ is not rooted in a member of $F$.

For $\mathcal{R}$, the product of any two functions is a function. Of course, this is true for $^*\mathcal{R}$ as well; i.e., the product of any two internal functions is an internal function.

9.6. EXAMPLE. We form the product of the internal function of Example 9.3 and the constant function $\{(t, \sqrt{(\omega/\pi)}) \mid t \in {}^*R\}$, which is an internal function. By the preceding observation, this yields the internal function

$$\{(t, \sqrt{\omega/\pi}\, \exp[-\omega t^2]) \mid t \in {}^*R\},$$

which we shall denote more simply by $\sqrt{\omega/\pi}\, \exp[-\omega x^2]$.

The function of Example 9.6 is a realization of the notion of a Dirac delta function; but there are other possible realizations. Generally an internal function, say $\delta$, is called a *Dirac delta function* if

(1) $\operatorname{dom}\delta = {}^*R$;

(2) $\forall t\, [t \not\simeq 0 \to \delta(t) \simeq 0]$;

(3) $\delta'(t) > 0$ for $t < 0$, and $\delta'(t) < 0$ for $t > 0$;

(4) $\int_{-\infty}^{\infty} \delta(t)\, \mathrm{d}t = 1$.

There might also be a requirement of infinite differentiability, except at specified points.

Here is another method of constructing members of $F$, expressed by the following statement:

(9.7) Corresponding to the standard real numbers $a$ and $b$, there is a function $f$ such that $f(t) = a$ for each $t \leqslant 0$, and $f(t) = b$ for each $t > 0$.

Since (9.7) can be formulated in the language of $\mathcal{R}$, and is true for $\mathcal{R}$, it is true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$.

9.8. EXAMPLE. Let $\epsilon$ be a nonzero infinitesimal, and let $f$ be the map of $^*R$ into $\{0, \epsilon\}$ such that $f(t) = 0$ for each $t \leqslant 0$, and $f(t) = \epsilon$ for each $t > 0$. By the preceding observation, $f \in {}^*F$. Clearly, $f$ is not rooted in any member of $F$.

We mention that each internal function yields two internal subsets of $^*R$, its domain and range. To see this, observe that the statements

$$(9.9)\qquad \forall f\, \exists S\, \forall x\, [x \in S \leftrightarrow \exists y\, [(x,y) \in f]],$$

$$(9.10)\qquad \forall f\, \exists S\, \forall y\, [y \in S \leftrightarrow \exists x\, [(x,y) \in f]]$$

are true for $\mathcal{R}$, where the quantifiers refer to $F$, $\mathcal{P}R$, $R$ and $R$. By the Transfer

Theorem, these statements are true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$. To interpret (9.9) and (9.10) in $^*\mathcal{R}$, we must realize that the quantifiers refer to $^*F$, $^*(\mathcal{P}R)$, $^*R$ and $^*R$. Let $g$ be any internal function; from (9.9), interpreted in $^*\mathcal{R}$, $\{x \mid (x,y) \in g\}$ is an internal subset of $^*\mathcal{R}$; from (9.10), interpreted in $^*\mathcal{R}$, $\{y \mid (x,y) \in g\}$ is an internal subset of $^*\mathcal{R}$. In other words, the domain and range of an internal function are internal subsets of $^*R$.

Notice that (9.9) and (9.10), interpreted in $^*\mathcal{R}$, are internal statements of $^*\mathcal{R}$. On the other hand, if $g$ is not rooted in a member of $F$, then

$$(9.11) \qquad \exists S\, \forall x\, [x \in S \leftrightarrow \exists y\, [(x,y) \in g]],$$

$$(9.12) \qquad \exists S\, \forall y\, [y \in S \leftrightarrow \exists x\, [(x,y) \in g]]$$

are external statements of $^*\mathcal{R}$. The point is that (9.11) and (9.12) cannot be interpreted in $\mathcal{R}$ since $g$ has no meaning there, even though $g$ is an internal entity of $^*\mathcal{R}$.

## 10. Continuity in $^*\mathcal{R}$

The fact that $^*F$ is a two-way extension of $F$ means that we face a double problem when deciding how to define continuity in $^*\mathcal{R}$. Here we are referring to *continuity of a function at a member of its domain*. In the case of a function rooted in a member of $F$, say $^*f$, we must define continuity so that for each $a \in R \cap \operatorname{dom} {}^*f$, $^*f$ is continuous at $a$ iff $f$ is continuous at $a$. The main problem is to decide how to extend the notion of continuity to members of the domain of $^*f$ in $^*R - R$.

Secondly, we must decide on the continuity of an internal function which is not rooted in a member of $F$. In this case there is no member of $F$ available to assist us.

We need a criterion for continuity in $^*\mathcal{R}$ that will handle both cases. Fortunately, the criteria for continuity in $\mathcal{R}$, discussed in Section 8, can be generalized to $^*\mathcal{R}$ in a straightforward manner.

First we shall generalize the Weierstrass criterion for continuity (8.1) to $^*\mathcal{R}$. Hereafter, by a *real* number we mean any member of $^*R$.

10.1. DEFINITION OF Q-CONTINUITY. We say that an internal function $f$ is Q-*continuous at* $a$, where $a \in \operatorname{dom} f$, if

$$(10.2) \qquad \forall \epsilon\, \exists \delta\, \forall x\, [\,|x - a| < \delta \rightarrow |f(x) - f(a)| < \epsilon]$$

is true for $^*\mathcal{R}$, where the Greek letters indicate quantification over positive real numbers and the Latin letter indicates quantification over $\operatorname{dom} f$.

Clearly, each constant function is Q-continuous at each $a \in {}^*R$, and the identity function is Q-continuous at each $a \in {}^*R$.

10.3. EXAMPLE. The internal function $\sin \omega x$ (see Example 9.4) is Q-continuous at each $a \in {}^*R$ since for $|x - a| < \delta$,

$$|\sin \omega x - \sin \omega a| = 2\,|\sin \tfrac{1}{2}\omega(x - a) \cos \tfrac{1}{2}\omega(x + a)|$$
$$< 2 \sin \tfrac{1}{2}\omega\delta,$$

provided that $\delta$ is chosen so that $\omega\delta$ is a positive infinitesimal and $\omega\delta < \epsilon$. But

$$2 \sin \tfrac{1}{2}\omega\delta < \omega\delta < \epsilon;$$

we conclude that $\sin \omega x$ is Q-continuous at each $a \in {}^*R$.

For our next example we consider an internal function of Example 9.8.

10.4. EXAMPLE. Let $\epsilon$ be a positive infinitesimal, and let $f$ be the function such that

$$f(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ \epsilon & \text{if } t > 0. \end{cases}$$

Certainly $f$ is Q-continuous at each $a \neq 0$, $a \in {}^*R$. However, $f$ is not Q-continuous at 0. In particular, there is no positive real number $\delta$ such that

$$\forall x\,[\,|x| < \delta \to |f(x)| < \tfrac{1}{2}\epsilon].$$

Indeed, for each positive real number $\delta$,

$$f(\tfrac{1}{2}\delta) = \epsilon.$$

Next we shall generalize the nonstandard criterion for continuity, i.e. (8.2). First notice that (8.2) can be expressed in terms of *monads* as

(10.5) $\qquad \forall x\,[x \in \mu(a) \to f(x) \in \mu(f(a))],$

where $\mu(t) = \{s \mid s \simeq t\}$, the monad of $t$, for each $t \in {}^*R$.

Now (10.5) is a statement about $^*\mathcal{R}$, and its quantifier refers to the domain of $f$. Moreover, $a$ is any member of the domain of $f$.

We shall call this sort of continuity *monadic* continuity, and we shall refer to it also as S-*continuity*.

10.6. DEFINITION OF S-CONTINUITY. We say that an internal function $f$ is S-*continuous at* $a$, where $a \in \operatorname{dom} f$, if

(10.7) $\quad \forall x\, [x \in \mu(a) \rightarrow f(x) \in \mu(f(a))]$

is true for $^*\mathcal{R}$, where the quantifier refers to the domain of $f$.

In other words, $f$ is S-continuous at $a$ provided that

(10.8) $\quad \forall x\, [x \simeq a \rightarrow f(x) \simeq f(a)]$

is true for $^*\mathcal{R}$.

Clearly each constant function is S-continuous at each $a \in {}^*R$, and the identity function is S-continuous at each $a \in {}^*R$.

Although the function $\sin \omega x$ of Example 10.3 is Q-continuous at 0, it is not S-continuous at 0. For $\pi/(2\omega) \in \mu(0)$, but $\sin \pi/(2\omega) \notin \mu(\sin 0) = \mu(0)$. Of course, this is a counter-example to the conjecture that each internal function that is Q-continuous at a member of its domain is also S-continuous there.

The internal function presented in Example 10.4 is a counter-example to the converse conjecture that each internal function that is S-continuous at a member of its domain, is also Q-continuous there. Clearly any internal function whose image contains only infinitesimals is S-continuous at each member of its domain. So the function of Example 10.4 is S-continuous at each $a \in {}^*R$.

Nonetheless, the concepts of Q-continuity and S-continuity are closely analogous. Indeed, the distinction between these notions reduces to the domain of two quantifiers in (10.2), the statement that characterizes Q-continuity at $a$.

10.9. LEMMA. *Let $f$ be any internal function, and let $a \in \operatorname{dom} f$. Then $f$ is* S-*continuous at $a$ iff*

(10.10) $\quad \forall \epsilon\, \exists \delta\, \forall x\, [|x-a| < \delta \rightarrow |f(x)-f(a)| < \epsilon]$

*is true for $^*\mathcal{R}$, where the Greek letters indicate quantification over the positive standard numbers, and the last quantifier refers to* $\operatorname{dom} f$.

*Proof.* (i) Assume that

$$\forall \epsilon\, \exists \delta\, \forall x\, [|x-a| < \delta \rightarrow |f(x)-f(a)| < \epsilon]$$

is true for $^*\mathcal{R}$, under the above agreements re quantifiers. Let $h$ be any positive standard number; by assumption there is a positive standard number, say $t$, such that

$$(10.11) \qquad \forall x\,[|x-a|<t \to |f(x)-f(a)|<h]$$

is true for $^*\mathcal{R}$. Take $x \in \mu(a)$; then $|x-a|<t$, so by (10.11), $|f(x)-f(a)|<h$. But $h$ is any positive standard number; thus $f(x) \simeq f(a)$. We conclude that $f$ is S-continuous at $a$. Notice the similarity, and the differences, between this argument and the first part of the proof of Lemma 8.3.

(ii) Assume that $f$ is S-continuous at $a$; i.e.,

$$\forall x\,[x \simeq a \to f(x) \simeq f(a)]$$

is true for $^*\mathcal{R}$, where the quantifier refers to dom $f$. Let $h$ be any positive standard number. We claim that there is a standard number $t$ such that

$$(10.12) \qquad \forall x\,[t>0 \wedge (|x-a|<t \to |f(x)-f(a)|<h)]$$

is true for $^*\mathcal{R}$, where the quantifier refers to dom $f$. Certainly each positive infinitesimal yields a true statement of $^*\mathcal{R}$ when put for $t$ throughout (10.12). If no standard number satisfies (10.12), it follows that the set of all positive infinitesimals is an internal subset of $^*R$. To see this, observe that

$$(10.13) \qquad \forall f\,\exists S\,\forall t\,[t \in S \leftrightarrow \forall x\,[t>0 \wedge (x \in \mathrm{dom} f \wedge |x-a|<t \to |f(x)-f(a)|<h)]]$$

is true for $\mathcal{R}$, where the first quantifier refers to $F$, the second to $\mathcal{P}R$, and the remaining quantifiers to $R$. Therefore (10.13) is true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$. In particular, since the function $f$ of our lemma is an internal function, there is an internal subset of $^*R$, say $S$, such that for each $t \in {}^*R$,

$$(10.14) \qquad t \in S \leftrightarrow \forall x\,[t>0 \wedge (x \in \mathrm{dom} f \wedge |x-a|<t \to |f(x)-f(a)|<h)]$$

is true for $^*\mathcal{R}$. If no standard number $t$ satisfies the RHS of (10.14), it follows that for each $t \in {}^*R$,

$$t \in S \quad \text{iff} \quad t>0 \wedge t \simeq 0;$$

i.e., $S$ is the set of all positive infinitesimals. So this set is an internal subset of $^*R$. Of course, it follows from this that the set of all infinitesimals is an internal subset of $^*R$. But this is impossible (see Section 1). This contradiction proves that (10.12) is true for $^*\mathcal{R}$, where $t$ is a positive standard number, and verifies our claim. We conclude that (10.10) is true for $^*\mathcal{R}$. This completes our proof of the lemma. □

In our next two lemmas we show that both Q-continuity and S-continuity are generalizations of the notion of continuity in $\mathcal{R}$.

10.15. LEMMA. *Let $f \in F$, and let $a \in$* dom$f$. *Then* $^*f$ *is* Q-*continuous at a iff f is continuous at a (in* $\mathcal{R}$*).*

*Proof.* By (8.1), $f$ is continuous at $a$ iff

$$(10.16) \qquad \forall\epsilon\, \exists\delta\, \forall x\, [|x-a| < \delta \rightarrow |f(x)-f(a)| < \epsilon]$$

is true for $\mathcal{R}$, where the Greek letters indicate quantification over positive standard numbers, and the Latin letter indicates quantification over dom$f$. It follows that (10.16) is true for $\mathcal{R}$ iff it is true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$. But (10.2) is the interpretation of (10.16) in $^*\mathcal{R}$. We conclude that $f$ is continuous at $a$ (in $\mathcal{R}$) iff $^*f$ is Q-continuous at $a$ (in $^*\mathcal{R}$). □

10.17. LEMMA. *Let $f \in F$, and let $a \in$* dom$f$. *Then* $^*f$ *is* S-*continuous at a iff f is continuous at a (in* $\mathcal{R}$*).*

*Proof.* By Lemma 8.3, $f$ is continuous at $a$ iff

$$\forall x\, [x \simeq a \rightarrow {}^*f(x) \simeq {}^*f(a)]$$

is true for $^*\mathcal{R}$, where the quantifier refers to dom $^*f$. In other words, $f$ is continuous at $a$ iff $^*f$ is S-continuous at $a$. □

Another approach, more in the spirit of Nonstandard Analysis, is to allow the extending process itself to make the decision as to which internal functions are to be labelled "continuous at $a$", where $a$ is a member of the domain of the function involved. To this purpose, we shall consider the relation of $\mathcal{R}$ that connects a function and a member of its domain, just in case the function is continuous at that member of its domain. Denoting this relation by PC (point continuity), we can appeal to $^*$PC, the corresponding relation of $^*\mathcal{R}$, to decide whether a given internal function $f$ is continuous at $a$, a member of its domain; i.e., we say that *$f$ is continuous at $a$* iff $(f, a) \in {}^*$PC.

10.18. LEMMA. *Let $f \in F$, and let $a \in$* dom$f$. *Then* $(^*f, a) \in {}^*$PC *iff f is continuous at a (in* $\mathcal{R}$*).*

*Proof.* By the Transfer Theorem, $(f, a) \in$ PC is true for $\mathcal{R}$ iff $(^*f, a) \in {}^*$PC is true for $^*\mathcal{R}$. □

It might appear that we are confronted with three competing notions of continuity at a number in $^*\mathcal{R}$. In fact there are only two since the notion of continuity represented by the relation $^*$PC is identical with our notion of Q-continuity at $a$.

10.19. LEMMA. *Let f be any internal function, and let* $a \in \text{dom} f$. *Then* $(f, a) \in {}^*\text{PC}$ *iff f is* Q-*continuous at a.*

*Proof.* Here we exploit the fact that Weierstrass characterized continuity in $\mathcal{R}$ by a statement in the language of $\mathcal{R}$. To be specific, for each function $f$ and for each member of its domain, say $a$,

$$(10.20) \qquad (f, a) \in \text{PC} \quad \text{iff} \quad \forall \epsilon\, \exists \delta\, \forall x\, [|x - a| < \delta \rightarrow |f(x) - f(a)| < \epsilon]$$

is true for $\mathcal{R}$. Therefore, by the Transfer Theorem (10.20) is true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$; i.e., for each internal function $f$ and for each member of its domain, say $a$,

$$(10.21) \qquad (f, a) \in {}^*\text{PC} \quad \text{iff} \quad f \text{ is Q-continuous at } a$$

since the RHS of (10.21) is the interpretation in $^*\mathcal{R}$ of the RHS of (10.20). □

Pursuing this approach, in which we use the extending process itself, let us move on to the notion of a *continuous* function, as opposed to the notion of continuity at a point (i.e. number). After all, we do possess a clear-cut notion of the continuous functions of $\mathcal{R}$; this can be formalized by an appropriate unary relation of $\mathcal{R}$, which we can call CF. Thus CF is the set of all functions, say $f$, such that $f$ is continuous at each member of its domain; of course, $\text{CF} \subset F$. Just as $F$ extends in a two-fold manner to $^*F$, so CF extends in a two-fold manner to $^*\text{CF}$. Since CF is a proper subset of $F$, just so $^*\text{CF}$ is a proper subset of $^*F$. The idea is to allow $^*\text{CF}$ to make the decision about the continuity of an internal function; thus we say that a member of $^*F$ is *continuous* iff it is also a member of $^*\text{CF}$.

To illustrate these ideas, consider the internal function $\sin \omega x$ of Example 10.3. Of course, $\sin \in \text{CF}$; moreover, for each standard number $t$ the composite function $\sin tx \in \text{CF}$. Therefore in $^*\mathcal{R}$ the composite function $\sin tx \in {}^*\text{CF}$ for each $t \in {}^*\mathcal{R}$. In particular, $\sin \omega x \in {}^*\text{CF}$; recall that $\sin \omega x$ is not S-continuous at 0 (see the remarks following Definition 10.6).

Next consider the internal function $f$ of Example 10.4. We want to decide whether $f$ is continuous, i.e., whether $f \in {}^*\text{CF}$. Observe that, for $\mathcal{R}$, no function whose domain is $R$ and whose range has exactly two members is a member of CF. Moreover, this fact can be expressed in the language of $\mathcal{R}$. Therefore no internal function whose domain is $^*R$ and whose range has exactly two members is a member of $^*\text{CF}$. We conclude that $f \notin {}^*\text{CF}$, i.e., $f$ is *not* continuous, even though $f$ is S-continuous at each member of its domain.

We pointed out in Lemma 10.19 that $(f, a) \in {}^*\text{PC}$ iff $f$ is Q-continuous at $a$. Just so, $f \in {}^*\text{CF}$ iff $f$ is Q-continuous at $a$ for each $a \in \text{dom} f$. Let us prove this.

10.22. LEMMA. *Let $f$ be any internal function. Then $f \in {}^*\mathrm{CF}$ iff $f$ is* Q-*continuous at $a$, for each $a \in$* dom $f$.

*Proof.* For each function $f$,

$$(10.23) \qquad f \in \mathrm{CF} \quad \text{iff} \quad \forall x\, [(f, x) \in \mathrm{PC}]$$

is true for $\mathcal{R}$, where the quantifier refers to dom $f$. Therefore this statement is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$; i.e., for each internal function $f$,

$$(10.24) \qquad f \in {}^*\mathrm{CF} \quad \text{iff} \quad \forall x\, [(f, x) \in {}^*\mathrm{PC}]$$

is true for ${}^*\mathcal{R}$, where the quantifier refers to dom $f$. By Lemma 10.19, $(f, x) \in {}^*\mathrm{PC}$ iff $f$ is Q-continuous at $x$. Thus, for each internal function $f$,

$$(10.25) \qquad f \in {}^*\mathrm{CF} \quad \text{iff} \quad \forall x\, [f \text{ is Q-continuous at } x]$$

is true for ${}^*\mathcal{R}$, where the quantifier refers to dom $f$. This completes our proof. □

Finally, we consider the problem of determining the derivative of an internal function. Obviously, the simplest procedure is to allow the extending process to make the decision for us. Let DF be the relation of $\mathcal{R}$ that pairs with each function its derivative (remember that the empty set is the derivative of certain functions, viz. those functions that are not differentiable); so $(f, g) \in \mathrm{DF}$ iff $g = f'$, the derivative of $f$. Now DF extends in a two-fold manner to ${}^*\mathrm{DF}$, a set of ordered pairs of internal functions, which is moreover a binary relation of ${}^*\mathcal{R}$. Of course, no two ordered pairs in ${}^*\mathrm{DF}$ have the same first term. So to determine the derivative of a given internal function, say $f$, we have only to consult ${}^*\mathrm{DF}$ as follows. Look up the ordered pair in ${}^*\mathrm{DF}$ whose first term is $f$; its second term is the derivative of $f$. Of course the familiar algebraic properties of derivatives in $\mathcal{R}$ are preserved in ${}^*\mathcal{R}$.

We mention that for each standard function $f$, $f'$ is defined as follows. Let $a \in R$. Then $a \in \operatorname{dom} f'$ iff

$${}^0\!\left(\frac{{}^*f(a+\epsilon) - {}^*f(a)}{\epsilon}\right) = {}^0\!\left(\frac{{}^*f(a+\delta) - {}^*f(a)}{\delta}\right)$$

whenever $\epsilon$ and $\delta$ are nonzero infinitesimals. If $a \in \operatorname{dom} f'$, then

$$f'(a) = {}^0\!\left(\frac{{}^*f(a+\epsilon) - {}^*f(a)}{\epsilon}\right),$$

where $\epsilon$ is any nonzero infinitesimal.

10.26. EXAMPLE. To determine the derivative of the internal function $\sin \omega x$ of Example 10.3, recall that $(\sin nx)' = n \cos nx$ for each standard natural number $n$. This is a fact about DF in $\mathcal{R}$ which can be expressed in the language of $\mathcal{R}$; so the corresponding statement is true about *DF in *$\mathcal{R}$, i.e., $(\sin nx)' = n \cos nx$ for each natural number $n$. In particular, $(\sin \omega x)' = \omega \cos \omega x$.

10.27. EXAMPLE. Let $f$ be the internal function of Example 10.4. Then $(f, g) \in {}^*\mathrm{DF}$, where

$$g = \{(t, 0) \mid t \in {}^*R \wedge t \neq 0\}.$$

This is so because each function in $\mathcal{R}$ of this sort (i.e., a union of two constant functions, and whose domain is $R$) has the zero function, with one pair deleted, for its derivative.

10.28. EXAMPLE. Consider a Dirac delta function $\delta = \sqrt{\kappa/\pi}\exp[-\kappa x^2]$, where $\kappa$ is any infinite natural number (see Example 9.6.) Clearly $\delta \in {}^*\mathrm{CF}$ and

$$\delta' = -2\kappa x\sqrt{\kappa/\pi}\exp[-\kappa x^2].$$

Notice that $\delta'(t) > 0$ if $t < 0$, and $\delta'(t) < 0$ if $t > 0$. Moreover, $\delta'(0) = 0$ and

$$\delta'(1/\kappa) = -2\sqrt{\kappa/\pi}\exp[-1/\kappa] \simeq -2\sqrt{\kappa/\pi},$$

which is negative and infinite; finally we mention that

$$\delta'(1) = -2\kappa\sqrt{\kappa/\pi}\exp[-\kappa] \simeq 0.$$

CHAPTER 3

# THE FIELD ${}^{\rho}\mathbb{R}$

## 1. Maximal ideals

A characteristic property of a maximal ideal, say $I$, of a commutative ring $\mathscr{A}$ with identity 1 is that the ring of cosets $A/I$ is a field. In fact, $A/I$ is a field iff $I$ is a maximal ideal of $\mathscr{A}$. We shall prove this statement here; this will prepare the way for our discussion of the nonarchimedean field ${}^{\rho}\mathbb{R}$ of Section 2.

First recall that a subset of $A$, say $I$, is an *ideal* of $\mathscr{A}$ provided that:

(1) $0 \in I$;

(2) $\forall xy\ [x,y \in I \to x - y \in I]$;

(3) $\forall ax\ [a \in A \wedge x \in I \to ax \in I]$.

An ideal $I$ of $\mathscr{A}$ is said to be a *proper* ideal if $I \neq A$ (i.e., $I$ is a proper subset of $A$).

An ideal $I$ of $\mathscr{A}$ is said to be *maximal* provided that $I$ is a proper ideal and is not a subset of any other proper ideal of $\mathscr{A}$.

1.1. LEMMA. *Let $\mathscr{A}$ be a commutative ring with identity* 1, *and let $\mathscr{A}$ have just two ideals*, $\{0\}$ *and* $A$. *Then $\mathscr{A}$ is a field.*

*Proof.* Notice that for each nonzero ring element $a$, $\{ax \mid x \in A\} \neq \{0\}$ is an ideal of $\mathscr{A}$. By assumption, $\{ax \mid x \in A\} = A$; so $\exists x\ [ax = 1]$. Thus each nonzero ring element has a multiplicative inverse; i.e., $\mathscr{A}$ is a field. □

1.2. LEMMA. *Let $\mathscr{A}$ be a commutative ring with identity* 1, *and let $I$ be a maximal ideal of $\mathscr{A}$. Then $A/I$ is a field.*

*Proof.* Clearly $A/I$ is a commutative ring with identity [1]. We shall show that $A/I$ has just two ideals, viz. $\{I\}$ and $A/I$. Notice that $I$ is the zero of $A/I$. Let $B$ be any ideal of $A/I$, $B \neq \{I\}$. It is easy to verify that

$$I' = \{x \in A \mid [x] \in B\}$$

is an ideal of $\mathcal{A}$. But $I \in B$ since $I$ is the zero of $A/I$; therefore $I \subset I'$. Since $I$ is maximal, either $I' = I$ or $I' = A$. By assumption, $B \neq \{I\}$; thus there is a nonzero ring element, say $[y]$, such that $[y] \in B$ and $[y] \neq I$. Therefore, $y \in I'$ and $y \notin I$. We conclude that $I' = A$, so $B = A/I$. This means that the ring $A/I$ has just two ideals, $\{I\}$ and $A/I$. Thus, by Lemma 1.1, $A/I$ is a field. □

1.3. LEMMA. *Let $\mathcal{A}$ be a commutative ring with identity* 1, *and let I be an ideal of $\mathcal{A}$ such that the ring of cosets $A/I$ is a field. Then I is maximal.*

*Proof.* Let $I'$ be an ideal of $\mathcal{A}$ such that $I \subset I'$, and suppose that there is a ring element $a$ such that $a \in I'$ and $a \notin I$. We shall show that $I' = A$. By assumption, $[a] \neq I$, so $[a]$ has a multiplicative inverse $[b]$; i.e.,

$$[a]\,[b] = [ab] = [1].$$

Let $x = ab - 1$. Then $x \in I$. Therefore $ab - x \in I'$ (since $x \in I'$ and $ab \in I'$); i.e., $1 \in I'$. Thus $I' = A$. We conclude that $I$ is maximal. □

## 2. The field $^{\rho}\mathcal{R}$

In a sense the field $\mathcal{R}$ suffers from the disadvantage of being archimedean, i.e., it has neither infinitely small nor infinitely large numbers. The field $^*\mathcal{R}$, on the other hand, suffers from a surfeit of infinitely small and infinitely large numbers. We propose in this chapter to construct and investigate a field midway between $\mathcal{R}$ and $^*\mathcal{R}$, which is nonarchimedean, but does not contain infinite numbers of unrestricted size, nor infinitesimals that are arbitrarily small.

The first step toward constructing this field consists in choosing a positive infinitesimal, say $\rho$. We shall call our field $^{\rho}\mathcal{R}$ since it is constructed from $^*\mathcal{R}$ and $\rho$. Recall our agreement that $^*\mathcal{R}$ is any sequentially comprehensive model of the postulate set of Section 2.1.

The next step in our construction is to form two subsets of $^*R$ which we call $M_0$ and $M_1$. $M_0$ consists of each number that is smaller, in absolute value, then $\rho^{-n}$ for some $n \in N$; i.e.,

$$M_0 = \{t \mid t \in {}^*R \text{ and } |t| < \rho^{-n} \text{ for some } n \in N\}.$$

$M_1$ consists of all numbers that are smaller than $\rho^n$ for each $n \in N$; i.e.,

$$M_1 = \{t \mid t \in {}^*R \text{ and } |t| < \rho^{n} \text{ for each } n \in N\}.$$

It is easy to show that both $M_0$ and $M_1$ form rings with respect to the addition and multiplication of ${}^*\mathcal{R}$. We claim that $M_1$ is a maximal ideal of the ring formed by $M_0$. To see this, let $x \in M_1$ and let $a \in M_0$. Then $|a| < \rho^{-n}$ for some $n \in N$, and $|x| < \rho^{n+m}$ for each $m \in N$. Thus $|ax| < \rho^m$ for each $m \in N$, so $ax \in M_1$. Moreover, $x \pm y \in M_1$ if $x, y \in M_1$. So $M_1$ is an ideal of $M_0$.

To prove that $M_1$ is maximal, we shall apply the argument used in Section 1.3. Let $a \in M_0 - M_1$ then for some $m, n \in N$,

$$\rho^m < |a| < \rho^{-n},$$

thus

$$\rho^n < |1/a| \leqslant \rho^{-m},$$

and it follows that $1/a \in M_0 - M_1$. Let $J$ be an ideal of the ring formed by $M_0$ such that $M_1$ is a proper subset of $J$, and let $a \in J - M_1$; then $a \in M_0 - M_1$. But we have just observed that $1/a \in M_0$; so $a \cdot (1/a) \in J$, i.e., $1 \in J$, and it follows that $J = M_0$. This establishes that $M_1$ is a maximal ideal of the ring formed by $M_0$.

Applying Lemma 1.2, we see that the quotient ring of residue classes $M_0/M_1$ is a field. This is the field that we call ${}^{\rho}\mathcal{R}$. Here $M_1 = [0]$ is the additive identity.

Next we introduce an order relation on our field ${}^{\rho}\mathcal{R}$; we shall follow the procedure discussed in Section 1.3, which centers on defining the *positive* elements of ${}^{\rho}R$ in terms of the positive elements of $M_0$ (with respect to the field ${}^*\mathcal{R}$). Let $[t]$ be a nonzero field element of ${}^{\rho}\mathcal{R}$; so $t \in M_0 - M_1$. Now all members of $[t]$ are positive in ${}^*\mathcal{R}$ or all members of $[t]$ are negative in ${}^*\mathcal{R}$. Accordingly, we define $[t]$ to be *positive* in ${}^{\rho}\mathcal{R}$ iff $t$ is positive in ${}^*\mathcal{R}$. Let $P'$ be the set of all positive members of ${}^{\rho}R$; then $P'$ has the following properties:

(1) $0 \notin P'$;

(2) $\forall x\, [x \neq 0 \rightarrow x \in P' \vee -x \in P']$;

(3) $\forall xy\, [x, y \in P' \rightarrow x + y \in P' \wedge xy \in P']$.

The idea is to use $P'$ to define a binary relation $<$ on ${}^{\rho}R$. We say that $x < y$ iff $x - y \in P'$. This binary relation is an order relation on ${}^{\rho}R$ and is compatible with both addition and multiplication (see Section 1.3). So ${}^{\rho}\mathcal{R}$ extends to an ordered field under $<$. We claim that this ordered field is nonarchimedean. It is enough to find $a \in P'$ such that $\forall n\, [na < [1]]$. Clearly $\rho \in M_0 - M_1$ and $\rho > 0$; so $[\rho] \in P'$. For each $n \in N$,

$$n[\rho] = \underbrace{[\rho] + \ldots + [\rho]}_{n\,[\rho]\text{'s}} = [n\rho].$$

But $n\rho < 1$ since $\rho$ is an infinitesimal of ${}^*\mathcal{R}$ (see Section 1.3); thus $1 - n\rho$ is positive in $^*\mathcal{R}$, so $[1 - n\rho] \in P'$ in $^\rho\mathcal{R}$. Therefore, by our definition of $<$ in $^\rho\mathcal{R}$, $[n\rho] < [1]$, i.e. $n[\rho] < [1]$. This proves that $^\rho\mathcal{R}$ is nonarchimedean and that $[\rho]$ is an infinitesimal of this field.

Now $\epsilon$ is an infinitesimal of $^\rho\mathcal{R}$ iff

$$\forall n\, [n\,|\epsilon| < [1]],$$

and $\kappa$ is infinite in $^\rho\mathcal{R}$ iff

$$\forall n\, [n\,[1] < |\kappa|].$$

Let $\epsilon \neq 0$; then $\epsilon$ is an infinitesimal of $^\rho\mathcal{R}$ iff $1/\epsilon$ is infinite in $^\rho\mathcal{R}$. As we have just observed, $[\rho]$ is an infinitesimal of $^\rho\mathcal{R}$, so $[1/\rho]$ is an infinite number of $^\rho\mathcal{R}$.

An important feature of our field $^\rho\mathcal{R}$ is that, putting it paradoxically, its infinite numbers are small and its infinitesimals are large. More precisely, the members of each infinite number are bounded above by $\rho^{-\kappa}$, and the members of each positive infinitesimal are bounded below by $\rho^{\kappa}$, where $\kappa$ is any member of $^*N - N$. Indeed, for each $\kappa \in {}^*N - N$ and for each $t \in M_0$,

$$[t] \neq 0 \rightarrow \rho^{\kappa} < |t| < \rho^{-\kappa}.$$

Notice that each coset $[t]$ is an interval in $M_0$, so in $^*R$; indeed,

$$[t] \subset \{x \,|\, x \in {}^*R \text{ and } t - \rho^n < x < t + \rho^n\}$$

for each $n \in N$, so $[t]$ is a subset of the intersection of all intervals of the form $[t - \rho^n, t + \rho^n]$.

Let us prove that $\mathcal{R}$ can be embedded in $^\rho\mathcal{R}$. Of course, $R \subset M_0$; moreover, 0 is the only standard number contained in $M_1$. Therefore each coset $[t]$ contains at most one standard number. If two cosets each contain a standard number, so do their sum and product. Accordingly we can identify each coset that contains a standard number $t$ with $t$. This "labelling" procedure preserves addition, multiplication and the order relation. So $\mathcal{R}$ is isomorphic to a subfield of $^\rho\mathcal{R}$; i.e., $\mathcal{R}$ is embedded in $^\rho\mathcal{R}$.

$^\rho\mathcal{R}$ is introduced in Robinson [1973].

## 3. Valuation

Let $t \in M_0 - M_1$, and let $i \in M_1$; so $[t] = [t + i]$. By assumption, there are standard natural numbers $m$ and $n$ such that

$$\rho^n \leqslant |t| < \rho^{-m}, \qquad \rho^n \leqslant |t + i| < \rho^{-m}.$$

Of course, if $0 < a < b$, then $\log_\rho b < \log_\rho a$; thus

$$-m < \log_\rho |t| < n, \qquad -m < \log_\rho |t + i| < n.$$

Clearly $\log_\rho |t|$ and $\log_\rho |t+i|$ are finite, so possess standard parts. We shall show that

$$\log_\rho |t| \simeq \log_\rho |t+i|.$$

Now the function $\log_\rho$ can be expressed in terms of the function ln as follows:

$$\log_\rho = \frac{\ln}{\ln \rho}.$$

So

$$\log_\rho |t+i| - \log_\rho |t| = \log_\rho \left| \frac{t+i}{t} \right| = \log_\rho |1 + i/t| = \frac{\ln |1 + i/t|}{\ln \rho}.$$

But $\ln \rho$ is infinite (and negative), and

$$\ln |1 + i/t| \simeq \ln 1 = 0$$

since $i/t \simeq 0$ and the function ln is continuous. Therefore

$$\log_\rho |t| \simeq \log_\rho |t+i|$$

and it follows that

$${}^0(\log_\rho |t|) = {}^0(\log_\rho |t+i|).$$

This proves that if $\alpha \in {}^\rho R$ and $\alpha \neq 0$, then

$${}^0(\log_\rho |x|) = {}^0(\log_\rho |y|),$$

provided that $x, y \in \alpha$.

In view of this fact, we define a map $v$ of ${}^\rho R$ into $R \cup \{\infty\}$ as follows. Let $v(0) = \infty$. If $\alpha \in {}^\rho R$ and $\alpha \neq 0$, let

$$v(\alpha) = {}^0(\log_\rho |x|),$$

where $x \in \alpha$. We claim that $v$ is a nonarchimedean valuation on ${}^\rho\mathcal{R}$; moreover, we shall prove that our field ${}^\rho\mathcal{R}$ is complete with respect to this valuation.

To illustrate our definition, notice that

$$v([\rho]) = 1,$$

$$v([\rho^n]) = n \quad \text{if } n \in R,$$

$$v([\rho^n]) = {}^0 n \quad \text{if } n \text{ is finite}.$$

Moreover, $\log_\rho t$ is a negative infinitesimal if $t$ is finite and $t > 1$; so ${}^0(\log_\rho t) = 0$;

thus for each such $t$ and for each $n \in N$,

$$v([t\rho^n]) = {}^0(\log_\rho |t\rho^n|) = {}^0(\log_\rho |t| + \log_\rho |\rho^n|) = {}^0(\log_\rho |t|) + n = n.$$

Our claim that $v$ is a nonarchimedean valuation on the field ${}^\rho\mathfrak{R}$ is based on the following two lemmas.

3.1. LEMMA. $\forall\alpha\beta\,[v(\alpha \cdot \beta) = v(\alpha) + v(\beta)]$, *where quantification is over* ${}^\rho R$.

*Proof.* If $\alpha = 0$ or $\beta = 0$, then $\alpha \cdot \beta = 0$, and $v(\alpha) = \infty$ or $v(\beta) = \infty$, so $v(\alpha \cdot \beta) = \infty$ and $v(\alpha) + v(\beta) = \infty$ (see the properties of $\infty$ listed in Section 1.4). Next assume that $\alpha \neq 0$ and $\beta \neq 0$, and let $\alpha = [x]$ and $\beta = [y]$. Then $\alpha \cdot \beta = [xy]$; so

$$\begin{aligned} v(\alpha \cdot \beta) = {}^0(\log_\rho |xy|) &= {}^0(\log_\rho |x| + \log_\rho |y|) \\ &= {}^0(\log_\rho |x|) + {}^0(\log_\rho |y|) \\ &= v(\alpha) + v(\beta). \end{aligned}$$

This completes our proof of the lemma. □

3.2. LEMMA. $\forall\alpha\beta\,[v(\alpha + \beta) \geqslant \min\{v(\alpha), v(\beta)\}]$, *where quantification is over* ${}^\rho R$.

*Proof.* If $\alpha + \beta = 0$, or if $\alpha = 0$, or if $\beta = 0$, then clearly

$$v(\alpha + \beta) \geqslant \min\{v(\alpha), v(\beta)\}.$$

Therefore we may assume that $\alpha + \beta \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$. Let $x \in \alpha$, and let $y \in \beta$; by the Trichotomy Law for ${}^*\mathfrak{R}$, $|x| \leqslant |y|$ or $|y| \leqslant |x|$. Since $\alpha + \beta = \beta + \alpha$, there is no loss of generality in restricting ourselves to the first case; so we shall assume that $|x| \leqslant |y|$. Thus

$$\log_\rho |y| \leqslant \log_\rho |x|,$$

so

$${}^0(\log_\rho |y|) \leqslant {}^0(\log_\rho |x|).$$

We shall prove that

$${}^0(\log_\rho |x + y|) \geqslant {}^0(\log_\rho |y|).$$

By the Triangle Inequality for ${}^*\mathfrak{R}$,

$$|x + y| \leqslant |x| + |y| \leqslant 2\,|y|$$

since $|x| \leqslant |y|$. Thus

$$\log_\rho |x + y| \geqslant \log_\rho |2y|,$$

so

$$^0(\log_\rho |x+y|) \geqslant {}^0(\log_\rho |2y|) = {}^0(\log_\rho 2 + \log_\rho |y|) = {}^0(\log_\rho 2) + {}^0(\log_\rho |y|).$$

But $\log_\rho 2$ is a negative infinitesimal, so $^0(\log_\rho 2) = 0$. Thus

$$^0(\log_\rho |x+y|) \geqslant {}^0(\log_\rho |y|),$$

i.e., $v(\alpha+\beta) \geqslant v(\beta)$. By assumption, $v(\beta) \leqslant v(\alpha)$; so

$$v(\alpha+\beta) \geqslant \min\{v(\alpha), v(\beta)\}.$$

This completes our proof of the lemma. □

Considering the requirements for a nonarchimedean valuation listed in Section 1.4, we conclude that $v$ is a nonarchimedean valuation on the field $^\rho R$. It follows that $v(-\alpha) = v(\alpha)$ and $v(1/\alpha) = -v(\alpha)$ if $\alpha \neq 0$ (see Section 1.4).

We shall prove that $^\rho R$ is complete with respect to the nonarchimedean valuation $v$ in the next section. There we shall need the following fact.

3.3. LEMMA. *Let $y \in {}^*R$, and let $x$ be a member of $M_0$ such that* $\log_\rho |x-y| > 0$. *Then $y \in M_0$.*

*Proof.* By assumption, $|x-y| = \rho^a$, where $a > 0$; so $x-y$ is finite. Moreover, there is a standard natural number $t$ such that $|x| < \rho^{-t}$. Thus

$$|y| \leqslant |y-x| + |x| < \rho^a + \rho^{-t} < 2\rho^{-t} < \rho^{-t-1},$$

so $y \in M_0$. □

Let us pin down our comment in the proof of Lemma 3.3 that $x-y$ is finite.

3.4. LEMMA. *Let $x \in {}^*R$, and let* $\log_\rho |x| > 0$; *then $x$ is finite.*

*Proof.* Let $|x| = \rho^a$; then $a > 0$. Now $\rho^t = \exp[t \ln \rho]$ for each $t \in {}^*R$. But $\ln \rho$ is negative, and the function exp is monotonically increasing; therefore the function $\rho^t$ is monotonically decreasing. Thus $\rho^a < \rho^{-a}$ since $a > 0$; so $(\rho^a)^2 < 1$, and we conclude that $0 < \rho^a < 1$. □

We shall need the following facts later.

3.5. LEMMA. *Let $a \in R$, where $a \neq 0$. Then* $v([a]) = 0$.

*Proof*. Let

$$|a| = \rho^t = \exp[t \ln \rho].$$

Now $\ln \rho$ is infinite and negative. Assume that $|a| > 1$; then $t \ln \rho > 0$ since $e^s > 1$ iff $s > 0$. So, if $|a| > 1$, then $t < 0$. Also, $t \ln \rho$ is finite since $\exp[t \ln \rho] = |a|$ is finite. Therefore $t \simeq 0$. This proves that $\log_\rho |a|$ is a negative infinitesimal in case $|a| > 1$. If $|a| = 1$, then $\log_\rho |a| = 0$, so $v([a]) = 0$. If $|a| < 1$, then

$$v([a]) = -v([1/a]) = 0$$

by the first part of this proof. We conclude that $v([a]) = 0$ if $a \in R$ and $a \neq 0$. □

3.6. LEMMA. *Let $x \simeq a$, where $a \in R$ and $a > 0$. Then $v([x]) = 0$.*

*Proof*. By assumption, $\frac{1}{2}a < x < \frac{3}{2}a$. Thus

$$\log_\rho(\tfrac{3}{2}a) < \log_\rho x < \log_\rho(\tfrac{1}{2}a),$$

so

$$^0(\log_\rho(\tfrac{3}{2}a)) \leqslant {}^0(\log_\rho x) \leqslant {}^0(\log_\rho(\tfrac{1}{2}a)),$$

i.e.,

$$v([\tfrac{3}{2}a]) \leqslant v([x]) \leqslant v([\tfrac{1}{2}a]).$$

Thus $v([x]) = 0$ by Lemma 3.5. □

3.7. COROLLARY. *Let $x \simeq a$, where $a \in R$ and $a \neq 0$. Then $v([x]) = 0$.*

*Proof*. Let $a < 0$. Then, by Lemma 3.6,

$$v([x]) = v([-x]) = 0$$

since $-x \simeq -a$. □

3.8. LEMMA. *Let $a, t \in R$, where $a \neq 0$. Then $v([a\rho^t]) = t$.*

*Proof*.

$$v([a\rho^t]) = v([a]) + v([\rho^t]) = v([\rho^t]) \quad \text{by Lemma 3.5}$$
$$= {}^0(\log_\rho |\rho^t|) = t. \ \square$$

Our next two lemmas relate the inequalities on ${}^\rho R$ and $R \cup \{\infty\}$.

3.9. LEMMA. *If $v(\beta) < v(\alpha)$, then $|\alpha| < |\beta|$.*

*Proof.* To avoid misunderstanding, we point out that

$$|\gamma| = \begin{cases} \gamma & \text{if } \gamma \geqslant 0, \\ -\gamma & \text{if } \gamma < 0. \end{cases}$$

If $v(\alpha) = \infty$, then $\alpha = 0$; but $\beta \neq 0$, so $|\alpha| < |\beta|$. Assume that $v(\alpha) \neq \infty$, i.e., $\alpha \neq 0$, and let $a \in |\alpha|$ and $b \in |\beta|$. Then

$$^0(\log_\rho b) < {^0(\log_\rho a)},$$

so $b = \rho^y$ and $a = \rho^x$, where $y < x$; thus $\rho^x < \rho^y$. Therefore $a < b$. This means that $|\alpha| < |\beta|$. □

3.10. LEMMA. *If* $|\alpha| < |\beta|$, *then* $v(\beta) \leqslant v(\alpha)$.

*Proof.* If $\alpha = 0$, then $v(\alpha) = \infty$; but $\beta \neq 0$, so $v(\beta) \in R$. Thus $v(\beta) < v(\alpha)$. Assume that $\alpha \neq 0$, and let $a \in |\alpha|$ and $b \in |\beta|$; then $v(\alpha) = {^0(\log_\rho a)}$ and $v(\beta) = {^0(\log_\rho b)}$. By assumption, $0 < a < b$; thus $\ln a < \ln b$, so

$$\ln a / \ln \rho > \ln b / \ln \rho;$$

i.e., $\log_\rho b < \log_\rho a$. Therefore

$$^0(\log_\rho b) \leqslant {^0(\log_\rho a)};$$

i.e., $v(\beta) \leqslant v(\alpha)$. □

## 4. Convergence

The role of $v$, the nonarchimedean valuation on $^{\rho}\mathcal{R}$ defined in Section 3, is to build up a metric $d$ on $^{\rho}\mathcal{R}$. As we saw in Section 1.5 this is achieved in two steps. First, we define the mapping $|\ |_v$, in terms of $v$, where

$$|\alpha|_v = e^{-v(\alpha)}$$

for each $\alpha \in {^{\rho}R}$; here we identify $e^{-\infty}$ with 0. This map has the seven properties listed in Section 1.5. Next we define the map $d$, in terms of $|\ |_v$, where

$$d(\alpha, \beta) = |\alpha - \beta|_v$$

for each $\alpha, \beta \in {^{\rho}R}$. In short, $d(\alpha, \beta) = e^{-v(\alpha - \beta)}$ for each $\alpha, \beta \in {^{\rho}R}$. As we pointed out in Section 1.5, $d$ is a metric on $^{\rho}R$ and satisfies the ultrametric inequality:

$$\forall \alpha\beta\gamma\, [d(\alpha, \gamma) \leqslant \max\{d(\alpha, \beta), d(\beta, \gamma)\}].$$

The metric $d$ allows us to define, in ${}^{\rho}\mathcal{R}$, the notions of a convergent sequence and its limit, and the notion of a Cauchy sequence. In the first place, a *sequence*, say $(\alpha_n)$, is a map of $N$ into ${}^{\rho}R$. Moreover, $(\alpha_n)$ converges to $\alpha$, where $\alpha \in {}^{\rho}R$, provided that the standard sequence $(d(\alpha_n, \alpha))$ converges to 0 (in $\mathcal{R}$). A sequence $(\alpha_n)$ is called a *Cauchy sequence* if

$$\forall \epsilon \, \exists n_0 \, \forall mn \, [m,n > n_0 \rightarrow d(\alpha_m, \alpha_n) < \epsilon],$$

where the first quantifier refers to the positive standard numbers, and the other quantifiers refer to $N$. If each Cauchy sequence converges in ${}^{\rho}\mathcal{R}$, we say that ${}^{\rho}\mathcal{R}$ is *complete* with respect to the nonarchimedean valuation $v$.

Notice that for each standard natural number $q$,

$$d(\alpha_m, \alpha_n) < \mathrm{e}^{-q} \quad \text{iff} \quad v(\alpha_m - \alpha_n) > q.$$

So $(\alpha_n)$ is a Cauchy sequence iff

$$\forall q \, \exists n_0 \, \forall mn \, [m,n > n_0 \rightarrow v(\alpha_m - \alpha_n) > q],$$

where the quantifiers refer to $N$.

In Section 1.5 we pointed out that sequences and series possess nice properties, from the viewpoint of convergence, if the metric is yielded by a nonarchimedean valuation and the field involved is complete with respect to that valuation (see Lemmas 1.5.7, 1.5.8, 1.5.11, 1.5.13 and 1.5.15). Accordingly, it is useful to establish the following fact.

4.1. THEOREM. *${}^{\rho}\mathcal{R}$ is complete with respect to $v$.*

*Proof.* Let $(\alpha_n)$ be any Cauchy sequence in ${}^{\rho}R$, and let $x_n \in \alpha_n$ for each $n \in N$. Then

$$\forall q \, \exists n_q \, \forall mn \, [m,n > n_q \rightarrow v(\alpha_m - \alpha_n) > q],$$

so

$$\forall q \, \exists n_q \, \forall mn \, [m,n > n_q \rightarrow {}^{0}(\log_\rho |x_m - x_n|) > q],$$

where the quantifiers refer to $N$. It follows that

$$\forall q \, \exists n_q \, \forall mn \, [m,n > n_q \rightarrow \log_\rho |x_m - x_n| > q].$$

Recall that ${}^{*}\mathcal{R}$ is sequentially comprehensive; so there is an internal sequence $(s_n)$, where $s_n \in {}^{*}R$ for each $n \in {}^{*}N$, such that $s_n = x_n$ for each $n \in N$. Therefore

$$\forall q \, \exists n_q \, \forall mn \, [m,n > n_q \rightarrow \log_\rho |s_m - s_n| > q] \tag{4.2}$$

is true for ${}^{*}\mathcal{R}$, where the quantifiers refer to $N$, and the sequence involved, namely $(s_n)$, is internal. In words, corresponding to each $q \in N$ there exists an $n_q \in N$ such that

$$m,n > n_q \to \log_\rho |s_m - s_n| > q,$$

where $m,n \in N$. Let $q \in N$, and choose $n_q$ in accordance with (4.2). Then

$$A = \{t \in {}^*N \mid \forall mn\, [m,n > n_q \wedge m,n \leqslant t \to \log_\rho |s_m - s_n| > q]\}$$

is an internal subset of ${}^*N$ that contains $N$. Thus, by the First Principle of Permanence 2.7.2, there is an infinite natural number $\kappa_q$ such that $t \in A$ for each $t \leqslant \kappa_q$, $t \in {}^*N$.

The idea is to choose $\kappa_0$ first, then $\kappa_1, \kappa_2$, and so on. In this way we are free to take $\kappa_1 < \kappa_0, \kappa_2 < \kappa_1, \kappa_3 < \kappa_2$, and so on. So $(\kappa_n)$ is a decreasing sequence, over $N$, of infinite natural numbers. Thus, by Lemma 2.7.7, there is an infinite natural number $\kappa$ such that $\kappa < \kappa_q$ for each $q \in N$. Therefore, for each $q \in N$, there is an $n_q \in N$ such that

$$m,n \leqslant \kappa \wedge m,n > n_q \to \log_\rho |s_m - s_n| > q,$$

where $m,n \in {}^*N$. Taking $n = \kappa$ yields

(4.3) $$m \leqslant \kappa \wedge m > n_q \to \log_\rho |s_m - s_\kappa| > q$$

for each $m \in {}^*N$. In (4.3) take $q = 0$ and $m = 1 + n_q$; then $\log_\rho |s_m - s_\kappa| > 0$, so by Lemma 3.3, $s_\kappa \in M_0$. Moreover, from (4.3),

$$\forall m\, [m \leqslant \kappa \wedge m > n_q \to {}^0(\log_\rho |s_m - s_\kappa|) \geqslant q],$$

where the quantifier refers to ${}^*N$; in particular,

(4.4) $$\forall m\, [m > n_q \to v([x_m - s_\kappa]) \geqslant q],$$

where the quantifier refers to $N$. We claim that $(\alpha_n)$ converges to $[s_\kappa]$. Given $q \in N$, choose $n_q$ in accordance with (4.2); for $m > n_q$,

$$d(\alpha_m, [s_\kappa]) = \exp(-v([x_m - s_\kappa])) \leqslant e^{-q}$$

by (4.4). Thus $(d(\alpha_n, [s_\kappa]))$ converges to 0. This proves that $(\alpha_n)$ converges to $[s_\kappa]$ and completes our proof of the theorem. □

Here are some basic facts about the map $|\ |_v$.

4.5. LEMMA. *If* $|\alpha| < |\beta|$, *then* $|\alpha|_v \leqslant |\beta|_v$.

*Proof.* By Lemma 3.10, $v(\beta) \leqslant v(\alpha)$; so $-v(\alpha) \leqslant -v(\beta)$. The function exp is monotonically increasing, so

$$e^{-v(\alpha)} \leqslant e^{-v(\beta)},$$

i.e., $|\alpha|_v \leqslant |\beta|_v$. □

4.6. LEMMA. *If* $|\alpha|_v < |\beta|_v$, *then* $|\alpha| < |\beta|$.

*Proof.* By assumption, $e^{-v(\alpha)} < e^{-v(\beta)}$; so $-v(\alpha) < -v(\beta)$, thus $v(\beta) < v(\alpha)$. Therefore $|\alpha| < |\beta|$ by Lemma 3.9. □

Our next lemma has a bearing on the discussion of continuity in Section 4.3.

4.7. LEMMA. *Let* $\alpha > 0$, *and let* $\gamma$ *be such that* $|\gamma - \alpha|_v < |\alpha|_v$. *Then* $\gamma > 0$.

*Proof.* By Lemma 4.6, $|\gamma - \alpha| < \alpha$; so $\gamma \neq 0$. Assume that $\gamma < 0$; then $-\gamma > 0$, so $\alpha - \gamma > \alpha$, thus $|\gamma - \alpha| > \alpha$. This contradiction proves that $\gamma > 0$. □

4.8. COROLLARY. *Let* $\alpha < 0$, *and let* $\gamma$ *be such that* $|\gamma - \alpha|_v < |\alpha|_v$. *Then* $\gamma < 0$.

*Proof.* By (2) of Section 1.5, $|\alpha - \gamma|_v < |-\alpha|_v$, where $-\alpha > 0$; i.e.,

$$|-\gamma - (-\alpha)|_v < |-\alpha|_v.$$

So, by Lemma 4.7, $-\gamma > 0$, i.e., $\gamma < 0$. □

Later we shall need the following fact.

4.9. LEMMA. *Let* $(\gamma_n)$ *be a sequence such that* $\lim(\gamma_n) = \gamma$, *and let* $\alpha_n$ *be between* $\gamma_n$ *and* $\gamma$ *for each* $n \in N$. *Then* $\lim(\alpha_n) = \gamma$.

*Proof.* Let $x_n \in \gamma_n$ and $t_n \in \alpha_n$ for each $n \in N$. Then $t_n$ is between $x_n$ and $x$ for each $n \in N$. Now, for each $n \in N$,

$$|\alpha_n - \gamma|_v = |[t_n - x]|_v \leqslant |[x_n - x]|_v$$

by Lemma 4.5, since $|t_n - x| \leqslant |x_n - x|$. So, for each $n \in N$,

$$|\alpha_n - \gamma|_v \leqslant |\gamma_n - \gamma|_v.$$

But $\lim(|\gamma_n - \gamma|_v) = 0$, so $\lim(|\alpha_n - \gamma|_v) = 0$; i.e., $\lim(\alpha_n) = \gamma$. □

## 5. Series

The notion of a series is discussed in Section 1.5 in a rather general setting. We now apply those ideas for the case of the field ${}^\rho\mathcal{R}$ and the metric induced by the nonarchimedean valuation $v$.

A series is an expression of the form $\Sigma_N \alpha_n$, where $\alpha_n \in {}^{\rho}R$ for each $n \in N$; the field elements $\alpha_n$ are generated by a map of $N$ into ${}^{\rho}R$. Intuitively, the formal expression $\Sigma_N \alpha_n$ represents the infinite sum

$$\alpha_0 + \alpha_1 + ... + \alpha_n + ...,$$

i.e., the sum of the terms of the infinite tuple $(\alpha_0, \alpha_1, ..., \alpha_n, ...)$ which has as many terms as there are natural numbers. We say that a series $\Sigma_N \alpha_n$ *converges* if the corresponding sequence of partial sums $(\sigma_n)$ converges, where

$$\sigma_n = \alpha_0 + ... + \alpha_n$$

for each $n \in N$. Throughout this discussion, our notion of convergence is based on the metric $d$ induced by our nonarchimedean valuation $v$.

If $(\sigma_n)$ converges, we say that the series $\Sigma_N \alpha_n$ converges to $\lim(\sigma_n)$, a member of ${}^{\rho}R$, and identify the formal expression $\Sigma_N \alpha_n$ with the field element $\lim(\sigma_n)$; i.e.,

$$\sum_N \alpha_n = \lim(\sigma_n).$$

The results of Section 1.5, which are of a general nature, apply in particular to our field ${}^{\rho}\mathcal{R}$. It is convenient to summarize these results here.

5.1. THEOREM. *If* $\lim(\alpha_n) = \alpha$, *then* $\lim(|\alpha_n|_v) = |\alpha|_v$.

5.2. LEMMA. $(\alpha_n)$ *is a Cauchy sequence iff* $\lim(\alpha_{n+1} - \alpha_n) = 0$.

5.3. LEMMA. $\Sigma_N \alpha_n$ *converges iff* $\lim(\alpha_n) = 0$.

5.4. LEMMA. $\Sigma_N \alpha_n$ *converges if* $\Sigma_N |\alpha_n|_v$ *converges. Recall that* $|\beta|_v = e^{-v(\beta)}$ *for each* $\beta \in {}^{\rho}R$.

5.5. LEMMA. $(\alpha_n)$ *converges iff* $\lim(\alpha_{n+1} - \alpha_n) = 0$.

5.6. LEMMA. *Let* $\lim(\alpha_n) = \alpha$, *and let B be a standard real number. Then*

$$\exists q \, \forall n \, [n > q \rightarrow v(\alpha_n) > B].$$

5.7. LEMMA. *Let* $\lim(\alpha_n) = \alpha$ *and* $\lim(\beta_n) = \beta$. *Then*

$$\lim(\alpha_n + \beta_n) = \alpha + \beta, \qquad \lim(\alpha_n \beta_n) = \alpha\beta.$$

5.8. LEMMA. *Let* $\Sigma_N \alpha_n = \alpha$ *and* $\Sigma_N \beta_n = \beta$. *Then*

$$\sum_N (\alpha_n + \beta_n) = \alpha + \beta.$$

5.9. LEMMA. *Let* $\Sigma_N \alpha_n = \alpha$, *let* $\Sigma_N \beta_n = \beta$, *and for each* $n \in N$ *let*

$$\gamma_n = \alpha_0\beta_n + \alpha_1\beta_{n-1} + \ldots + \alpha_n\beta_0.$$

*Then*

$$\sum_N \gamma_n = \alpha\beta.$$

5.10. LEMMA. *Let* $(\alpha_n)$ *be a convergent sequence such that* $v(\alpha_n) = t$ *for each* $n > j$, *where* $t \in R$ *and* $j \in N$. *Then* $\lim(\alpha_n) \neq 0$.

5.11. THEOREM. *Let* $\lim(\alpha_n) = \alpha$, *and let j be a standard natural number such that* $v(\alpha_m) = t$ *for each* $m > j$. *Then* $v(\alpha) = t$.

5.12. LEMMA. $\Sigma_N \alpha_n$ *converges iff* $\lim(v(\alpha_n)) = \infty$; *i.e.*,

$$\forall B\, \exists q\, \forall n\, [n > q \to v(\alpha_n) > B].$$

5.13. THEOREM. *Let* $\lim(\alpha_n) = \alpha$, *where* $\alpha \neq 0$. *Then*

$$\exists j\, \forall m\, [m > j \to |\alpha_m|_v = |\alpha|_v].$$

To dispell the notion that each series converges, we point out that by Lemma 5.3, $\Sigma_N \alpha_n$ converges iff $\lim(\alpha_n) = 0$. If there is a standard natural number $j$ such that $\alpha_n = \alpha_j$ for each $n \geqslant j$ and $\alpha_j \neq 0$, then $\lim(\alpha_n) = \alpha_j \neq 0$. So $\Sigma_N \alpha_n$ diverges. For example, let $\alpha$ be the map of $N$ into ${}^\rho R$ such that

$$\alpha(n) = \begin{cases} 0 & \text{if } n < 5, \\ [1] & \text{if } n \geqslant 5, \end{cases}$$

then $\lim(\alpha_n) = [1]$; so $\Sigma_N \alpha_n$ diverges. Indeed, the sequence of partial sums is $(0, 0, 0, 0, 0, [1], [2], [3], \ldots)$ which converges iff $([n])$ converges. By Lemma 5.2, this sequence diverges.

## 6. $\rho$-series

Each series of the form $\Sigma_N a_n [\rho]^{\nu_n}$, where $a_n, \nu_n \in R$ for each $n \in N$, $\nu_0 < \nu_1 < \nu_2 < \ldots$, and $\{\nu_n \mid n \in N\}$ is unbounded, is called a *$\rho$-series*.

Here are some facts about $\rho$-series.

6.1. LEMMA. *Each $\rho$-series converges.*

*Proof.* Let $\Sigma_N a_n [\rho]^{\nu_n}$ be any $\rho$-series, and let $\alpha_n = [a_n \rho^{\nu_n}]$ for each $n \in N$. If $a_n \neq 0$, then $v(\alpha_n) = \nu_n$ by Lemma 3.8; of course, if $a_n = 0$, then $v(\alpha_n) = \infty$. Thus $v(\alpha_n) \geqslant \nu_n$ for each $n \in N$. By Lemma 5.12, $\Sigma_N \alpha_n$ converges iff $(v(\alpha_n))$ increases without bound iff $(\nu_n)$ is unbounded. But the sequence $(\nu_n)$ is unbounded by assumption. We conclude that each $\rho$-series converges. □

6.2. COROLLARY. *Let $\Sigma_N a_n [\rho]^{\nu_n}$ be any $\rho$-series; let $(\sigma_n)$ be the corresponding sequence of partial sums, i.e.,*

$$\sigma_n = a_0 [\rho]^{\nu_0} + \ldots + a_n [\rho]^{\nu_n}$$

*for each $n \in N$. Then $(\sigma_n)$ converges and*

$$\lim(\sigma_n) = \sum_N a_n [\rho]^{\nu_n}.$$

6.3. LEMMA. *Let $\Sigma_N a_n [\rho]^{\nu_n}$ be any $\rho$-series, and let $(\sigma_n)$ be the corresponding sequence of partial sums. Either $a_n = 0$ for each $n \in N$, or there is a standard natural number $j$ such that $v(\sigma_n) = \nu_j$ for each $n \geqslant j$.*

*Proof.* If $a_n \neq 0$ for some $n \in N$, then there is a smallest standard natural number $j$ such that $a_j \neq 0$. For $n \geqslant j$,

$$\sigma_n = [a_0 \rho^{\nu_0} + \ldots + a_n \rho^{\nu_n}] = [a_j \rho^{\nu_j} + \ldots + a_n \rho^{\nu_n}],$$

thus

$$\begin{aligned} v(\sigma_n) &= {}^0(\log_\rho |a_j \rho^{\nu_j} + \ldots + a_n \rho^{\nu_n}|) \\ &= {}^0(\log_\rho |\rho^{\nu_j}(a_j + a_{j+1}\rho^{\nu_{j+1}-\nu_j} + \ldots + a_n \rho^{\nu_n - \nu_j})|) \\ &= \nu_j + {}^0(\log_\rho |a_j + a_{j+1}\rho^{\nu_{j+1}-\nu_j} + \ldots + a_n \rho^{\nu_n - \nu_j}|). \end{aligned}$$

But

$$a_{j+1}\rho^{\nu_{j+1}-\nu_j} + \ldots + a_n \rho^{\nu_n - \nu_j} \simeq 0$$

and $a_j \neq 0$; so, by Corollary 3.7,

$$v([a_j + a_{j+1}\rho^{\nu_{j+1}-\nu_j} + \ldots + a_n \rho^{\nu_n - \nu_j}]) = 0.$$

Thus $v(\sigma_n) = \nu_j$. This completes our proof of the Lemma. □

Alternatively, we can establish this lemma by applying Lemma 1.4.3, since

$$v([a_j \rho^{\nu_j}]) < v(\sigma_n - \sigma_j)$$

for each $n > j$.

6.4. COROLLARY. *Let $\Sigma_N a_n [\rho]^{\nu_n}$ be any ρ-series such that some $a_n \neq 0$. Then $\Sigma_N a_n [\rho]^{\nu_n} \neq 0$.*

*Proof.* Let $(\sigma_n)$ be the corresponding sequence of partial sums. By Lemma 6.3, there is a standard natural number $j$ such that $v(\sigma_n) = \nu_j$ for each $n \geqslant j$. Therefore, by Lemma 5.10, $\lim(\sigma_n) \neq 0$; thus $\Sigma_N a_n [\rho]^{\nu_n} \neq 0$. □

6.5. LEMMA. *Let $\Sigma_N a_n [\rho]^{\nu_n}$ be any ρ-series. Then $\Sigma_N a_n [\rho]^{\nu_n} = 0$ iff $a_n = 0$ for each $n \in N$.*

*Proof.* If $a_n = 0$ for each $n \in N$, then $\sigma_n = 0$ for each $n$; so $\lim(\sigma_n) = 0$ and $\Sigma_N a_n [\rho]^{\nu_n} = 0$. If $a_n \neq 0$ for some $n \in N$, then $\Sigma_N a_n [\rho]^{\nu_n} \neq 0$ by Corollary 6.4. □

The following lemma has a bearing on the relation between ρ-series and the generalized power series of Section 1.7.

6.6. LEMMA. *Let $\Sigma_N a_n [\rho]^{\nu_n}$ be a ρ-series such that some $a_n \neq 0$. Then*

$$v(\sum_N a_n [\rho]^{\nu_n}) = \nu_j,$$

*where $j$ is the smallest standard natural number such that $a_j \neq 0$.*

*Proof.* Let $(\sigma_n)$ be the corresponding sequence of partial sums. By Corollary 6.4, $\Sigma_N a_n [\rho]^{\nu_n} \neq 0$; thus $\lim(\sigma_n) \neq 0$. For each $n \geqslant j$, $v(\sigma_n) = \nu_j$, by Lemma 6.3, where $j$ is the smallest standard natural number such that $a_j \neq 0$. Therefore, by Theorem 5.11, $v(\lim(\sigma_n)) = \nu_j$; i.e.,

$$v(\sum_N a_n [\rho]^{\nu_n}) = \nu_j. \square$$

As a simplifying convention, we shall identify two ρ-series if they yield the same series upon suppressing each term whose coefficient is zero, or if each coefficient of each series is zero.

Next we want to relate the definition of $<$ for ${}^\rho\mathcal{R}$ (see Section 2) to the definition of $<$ for $\mathcal{L}$ (see Section 1.7).

6.7. LEMMA. *Let* $\Sigma_N a_n [\rho]^{\nu_n}$ *be a* $\rho$*-series such that* $a_0 \neq 0$. *Then* $\Sigma_N a_n [\rho]^{\nu_n} > 0$ *iff* $a_0 > 0$.

*Proof*. Let $x \in \Sigma_N a_{n+1} [\rho]^{\nu_{n+1}}$; then

$$\sum_N a_n [\rho]^{\nu_n} = [a_0 \rho^{\nu_0} + x].$$

Now $v([x]) \geqslant \nu_1 > \nu_0$ (indeed, $v([x]) = \nu_1$ if $a_1 \neq 0$); so

$$v([x]) > v([a_0 \rho^{\nu_0}]),$$

thus $|x| < |a_0 \rho^{\nu_0}|$. Therefore

$$a_0 \rho^{\nu_0} + x > 0 \quad (\text{in } {}^*\mathbb{R}) \quad \text{iff} \quad a_0 > 0,$$

so

$$[a_0 \rho^{\nu_0} + x] > 0 \quad (\text{in } {}^{\rho}\mathbb{R}) \quad \text{iff} \quad a_0 > 0,$$

i.e.,

$$\sum_N a_n [\rho]^{\nu_n} > 0 \quad \text{iff} \quad a_0 > 0. \ \square$$

Our next two lemmas show the connection between addition and multiplication in ${}^{\rho}\mathbb{R}$, restricted to $\rho$-series, and the corresponding operations of the field $\mathcal{L}$.

6.8. LEMMA. $\Sigma_N a_n [\rho]^{\nu_n} + \Sigma_N b_n [\rho]^{\mu_n} = \Sigma_N c_n [\rho]^{\lambda_n}$, *provided the series on the LHS are* $\rho$*-series, and the series on the RHS is obtained from these* $\rho$*-series by applying the definition for the sum of two generalized power series (see Section* 1.7).

*Proof*. Consider the manner in which $\Sigma_N c_n [\rho]^{\lambda_n}$ is obtained from $\Sigma_N a_n [\rho]^{\nu_n}$ and $\Sigma_N b_n [\rho]^{\mu_n}$. For each $n \in N$ there is a corresponding $m \in N$ $(n \leqslant m \leqslant 2n+1)$ such that

$$[a_0 \rho^{\nu_0} + \ldots + a_n \rho^{\nu_n}] + [b_0 \rho^{\mu_0} + \ldots + b_n \rho^{\mu_n}] = [c_0 \rho^{\lambda_0} + \ldots + c_m \rho^{\lambda_m}].$$

Therefore, by Lemma 5.9,

$$\lim([a_0 \rho^{\nu_0} + \ldots + a_n \rho^{\nu_n}]) + \lim([b_0 \rho^{\mu_0} + \ldots + b_n \rho^{\mu_n}]) = \lim([c_0 \rho^{\lambda_0} + \ldots + c_n \rho^{\lambda_n}]),$$

i.e.,

$$\sum_N a_n [\rho]^{\nu_n} + \sum_N b_n [\rho]^{\mu_n} = \sum_N c_n [\rho]^{\lambda_n}. \ \square$$

6.9. LEMMA. $(\Sigma_N a_n [\rho]^{\nu n}) \cdot (\Sigma_N b_n [\rho]^{\mu n}) = \Sigma_N c_n [\rho]^{\lambda n}$, *provided the series on the LHS are ρ-series, and the series on the RHS is obtained from these ρ-series by applying the definition for the product of two generalized power series (see Section* 1.7).

*Proof.* Consider the manner in which $\Sigma_N c_n [\rho]^{\lambda n}$ is obtained from $\Sigma_N a_n [\rho]^{\nu n}$ and $\Sigma_N b_n [\rho]^{\mu n}$. For each $n \in N$ there is a corresponding $m \in N$ $(n \leqslant m < (n+1)^2)$ such that

$$[a_0\rho^{\nu_0} + ... + a_n\rho^{\nu n}]\,[b_0\rho^{\mu_0} + ... + b_n\rho^{\mu n}] = [c_0\rho^{\lambda_0} + ... + c_m\rho^{\lambda m}].$$

Therefore, by Lemma 5.9,

$$(\lim([a_0\rho^{\nu_0} + ... + a_n\rho^{\nu n}])) \cdot (\lim([b_0\rho^{\mu_0} + ... + b_n\rho^{\mu n}])) = \lim([c_0\rho^{\lambda_0} + ... + c_n\rho^{\lambda n}]),$$

i.e.,

$$(\sum_N a_n [\rho]^{\nu n}) \cdot (\sum_N b_n [\rho]^{\mu n}) = \sum_N c_n [\rho]^{\lambda n}. \square$$

We have shown that ρ-series behave under the field operations of $^\rho\mathcal{R}$ in the same way as the generalized power series of the field $\mathcal{L}$. To be specific, let Φ be the map of $L$ into $^\rho R$ such that

$$\Phi(\sum_N a_n t^{\nu n}) = \sum_N a_n [\rho]^{\nu n}$$

for each generalized power series $\Sigma_N a_n t^{\nu n} \in L$. Then Φ is a homomorphism of $\mathcal{L}$ into $^\rho\mathcal{R}$. Notice that Φ is a one–one map since $\Phi(\Sigma_N a_n t^{\nu n}) = 0$ (the additive identity of $^\rho\mathcal{R}$) iff $a_n = 0$ for each $n \in N$; i.e., $\Phi(\Sigma_N a_n t^{\nu n}) = 0$ iff $\Sigma_N a_n t^{\nu n} = 0$ (the additive identity of $\mathcal{L}$). Therefore $\mathcal{L}$ is isomorphic to $\Phi\mathcal{L}$, a subfield of $^\rho\mathcal{R}$. From Lemma 6.7, the map Φ preserves the order relation of $\mathcal{L}$; indeed, for each ρ-series $\Sigma_N a_n [\rho]^{\nu n}$,

$$\sum_N a_n [\rho]^{\nu n} \text{ is positive (in } {}^\rho\mathcal{R}) \quad \text{iff} \quad \sum_N a_n t^{\nu n} \text{ is positive (in } \mathcal{L}),$$

so

$$\sum_N a_n [\rho]^{\nu n} < \sum_N b_n [\rho]^{\mu n} \quad \text{iff} \quad \sum_N a_n t^{\nu n} < \sum_N b_n t^{\mu n}.$$

Moreover, for each ρ-series $\Sigma_N a_n [\rho]^{\nu n}$,

$$v(\sum_N a_n [\rho]^{\nu n}) = v(\sum_N a_n t^{\nu n})$$

by Lemma 6.6. Thus Φ is *analytic* (i.e., value preserving).

We conclude that the system $(^\rho\mathcal{R}, v)$ is an extension of the system $(\mathcal{L}, v)$.

## 7. Iotas and megas

By construction, the field $^\rho\mathcal{R}$ lies between the fields $\mathcal{R}$ and $^*\mathcal{R}$; indeed $^\rho\mathcal{R}$ is an extension of $\mathcal{R}$ that includes large infinitesimals and small infinite numbers, but excludes small infinitesimals and large infinite numbers. As we know, we can study the real number system $\mathcal{R}$ effectively by utilizing various concepts of $^*\mathcal{R}$; e.g., infinitesimals, infinite natural numbers, infinite sums. Similarly, certain concepts of $^*\mathcal{R}$ assist our study of $^\rho\mathcal{R}$; e.g., the concept of a small infinitesimal and the concept of a large infinite number.

Intuitively, we regard each member of $M_1$, say $i$, as a *small* infinitesimal, and its multiplicative inverse (provided that $i \neq 0$) as a *large* infinite number. Introducing some terminology, we shall call each member of $M_1$ an *iota*, and we shall call the multiplicative inverse of each nonzero iota a *mega*. We point out that $i \in {}^*R$ is an iota iff $|i| < \rho^n$ for each $n \in N$; also, $\kappa \in {}^*R$ is a mega iff $|\kappa| > \rho^{-n}$ for each $n \in N$. So $\kappa$ is a mega iff $\kappa \in {}^*R - M_0$.

Let $\approx$ be the binary relation on $^*R$ such that $x \approx y$ (read *x is infinitely close to y*) iff $x - y$ is an iota. In particular, if $y \in M_0$, then $x \approx y$ iff $x \in [y]$, a member of $^\rho R$.

7.1. LEMMA. *$\approx$ is an equivalence relation on $^*R$.*

*Proof.* Use the fact that $M_1$ is a ring. □

One goal in constructing $^\rho\mathcal{R}$ is to raise the large infinitesimals and small infinite numbers of $^*\mathcal{R}$ to the status of standard numbers; more precisely, we are referring to the equivalence classes that contain large infinitesimals or small infinite numbers. To a certain extent, this allows us to think of large infinitesimals and small infinite numbers in the same way that we think of standard numbers. Accordingly, we shall use the iotas and megas of $^*\mathcal{R}$ to study $^\rho\mathcal{R}$ in the same manner that infinitesimals and infinite numbers assist our study of $\mathcal{R}$. Just as for infinitesimals, notice that the equivalence relation $\approx$ provides us with a convenient abbreviation for the statement "$i$ is an iota", namely "$i \approx 0$".

The arithmetic of iotas is similar to the arithmetic of infinitesimals. For example, the sum of two iotas is an iota, and the product of two iotas is an iota.

Here is a useful way of characterizing iotas.

7.2. CRITERION FOR IOTAS. *$i$ is an iota iff there is an infinite natural number, say $\nu$, such that $|i| < \rho^\nu$.*

*Proof*. Let $i \in {}^*R$.

(i) Assume that $i \approx 0$. Then $|i| < \rho^n$ for each $n \in N$. Notice that

$$\{n \in {}^*N \mid |i| < \rho^n\}$$

is an internal subset of ${}^*N$ that contains $N$. Therefore, by the First Principle of Permanence 2.7.2, there is an infinite natural number, say $\nu$, such that $|i| < \rho^\nu$.

(ii) Assume that $|i| < \rho^\nu$ for some $\nu \in {}^*N - N$. For each $n \in N$, $\rho^\nu < \rho^n$; thus $|i| < \rho^n$ for each $n \in N$. So $i \in M_1$, i.e., $i \approx 0$. □

Here are some applications of our Criterion for Iotas.

7.3. LEMMA. *If i is an iota and a is not a mega, then ai is an iota.*

*Proof*. We are given that $|i| < \rho^\nu$ for some $\nu \in {}^*N - N$, and that $|a| < \rho^{-n}$ for some $n \in N$. Thus $|ai| < \rho^{\nu - n}$. But $\nu - n \in {}^*N - N$; so, by the Criterion for Iotas, $ai$ is an iota. □

7.4. COROLLARY. *If $\kappa$ is a mega and if a is not an iota, then $a\kappa$ is a mega.*

*Proof*. We are given that $1/\kappa$ is an iota and that $1/a$ is not a mega. Thus, by Lemma 7.3, $(1/a)\cdot(1/\kappa) \approx 0$; so $a\kappa$ is a mega. □

Later we shall need the following facts.

7.5. LEMMA. $(1-h)^\kappa \approx 0$, *provided that*:

(i) *$\kappa$ is a mega and $\kappa \in {}^*N$,*

(ii) $0 < h \leqslant 1$,

(iii) *h is not an iota.*

*Proof*. For $\mathcal{R}$,

$$(1-a)^n \leqslant 1/(1+an)$$

if $n \in N$ and $0 < a \leqslant 1$. Therefore, for ${}^*\mathcal{R}$,

$$(1-h)^\kappa \leqslant 1/(1+h\kappa).$$

But $1 + h\kappa$ is a mega; so $1/(1+h\kappa) \approx 0$. Applying the Criterion for Iotas we conclude that $(1-h)^\kappa \approx 0$. □

7.6. LEMMA. $h^\kappa/\kappa! \approx 0$ *if $\kappa$ is a mega, $\kappa \in {}^*N$, and h is finite.*

*Proof.* Since $h$ is not infinite, $|h| < n$ for some $n \in N$. Now $n^\nu/\nu! \simeq 0$ if $\nu \in {}^*N - N$. Therefore,

$$n^{\kappa-1}/(\kappa-1)! < 1;$$

so $n^\kappa/\kappa! < n/\kappa$. By Lemma 7.3, $n/\kappa \approx 0$. Applying the Criterion for Iotas, $n^\kappa/\kappa! \approx 0$. But

$$0 < |h|^\kappa/\kappa! < n^\kappa/\kappa!;$$

applying the Criterion for Iotas again, we conclude that $h^\kappa/\kappa!$ is an iota. □

We can strengthen Lemma 7.6.

7.7. LEMMA. $h^\kappa/\kappa! \approx 0$ *if* $\kappa$ *is a mega,* $\kappa \in {}^*N$, *and* $h$ *is not a mega.*

*Proof.* We are given that $|h| < \rho^{-m}$ for some $m \in N$. Thus

$$\begin{aligned}
0 < \frac{|h|^\kappa}{\kappa!} &< \frac{(\rho^{-m})^\kappa}{\kappa!} \\
&= \frac{1}{\kappa\rho^m}\,\frac{1}{(\kappa-1)\rho^m}\cdots\frac{1}{2\rho^m}\,\frac{1}{\rho^m} \\
&= \frac{1}{\kappa\rho^{2m}}\cdots\frac{1}{(\kappa-\nu+1)\rho^{2m}}\,\frac{1}{(\kappa-\nu)\rho^m}\cdots\frac{1}{(\nu+1)\rho^m}\,\frac{1}{\nu}\,\frac{1}{\nu-1}\cdots\frac{1}{1} \\
&< \frac{1}{\kappa\rho^{2m}}\cdots\frac{1}{(\kappa-\nu+1)\rho^{2m}}\,\frac{1}{(\kappa-\nu)\rho^m}\cdots\frac{1}{(\nu+1)\rho^m} \\
&< \frac{1}{\kappa\rho^{2m}}\cdots\frac{1}{(\kappa-\nu+1)\rho^{2m}} \approx 0,
\end{aligned}$$

where $\nu$ is the smallest integer greater than $\rho^{-m}$. So $h^\kappa/\kappa! \approx 0$. □

Our proofs of the preceding lemmas have used the fact that a number is an iota if it is smaller, in absolute value, than some iota. Of course, this is a basic property of iotas and deserves a proof.

7.8. LEMMA. *Let* $i \approx 0$, *and let* $|h| < |i|$, *where* $h \in {}^*R$. *Then* $h \approx 0$.

*Proof.* By the Criterion for Iotas, $|i| < \rho^\nu$ for some $\nu \in {}^*N - N$. Therefore $|h| < \rho^\nu$; so $h$ is an iota by the Criterion for Iotas. □

Similarly, a number is a mega if it is larger, in absolute value, than some mega.

7.9. LEMMA. *Let $\kappa$ be any mega, and let $|h| > |\kappa|$, where $h \in {}^*R$. Then $h$ is a mega.*

*Proof.* $1/\kappa \approx 0$ and $|1/h| < |1/\kappa|$; by Lemma 7.8, $1/h \approx 0$. We conclude that $h$ is a mega. □

Later we shall need the following principle of permanence, which asserts that each natural number greater than some $q \in M_0$ has a specified internal property, provided that each mega in ${}^*N$ has the property. This is analogous to the Second Principle of Permanence 2.7.3.

7.10. FOURTH PRINCIPLE OF PERMANENCE. *Let $A$ be an internal subset of ${}^*N$ such that $\kappa \in A$ if $\kappa$ is a mega in ${}^*N$. Then there is an infinite natural number in $M_0$, say $q$, such that*

$$\forall n\, [n \in {}^*N \wedge n > q \rightarrow n \in A].$$

*Proof.* Either each infinite natural number is in $A$, or these is an infinite natural number $\kappa$ such that $\kappa \notin A$. If the former, there is nothing to prove. If the latter, then ${}^*N - A$ is a nonempty internal subset of ${}^*N$ which is bounded above. Therefore ${}^*N - A$ has a greatest member, say $q$. So

$$\forall n\, [n \in {}^*N \wedge n > q \rightarrow n \in A]. \square$$

The following principle of permanence is analogous to the Third Principle of Permanence 2.7.5.

7.11. FIFTH PRINCIPLE OF PERMANENCE. *Let $(i_n)$ be an internal sequence, in ${}^*\mathcal{R}$, such that $i_n \approx 0$ for each $n \in N$. Then there is an infinite natural number $\kappa$ such that $i_n \approx 0$ for each $n < \kappa$, $n \in {}^*N$.*

*Proof.* Notice that $(\rho^{-n} i_n)$ is an internal sequence of ${}^*\mathcal{R}$. For each $n \in N$, $\rho^{-n} i_n \approx 0$; so $|\rho^{-n} i_n| < 1$. By Lemma 2.7.4, there is an infinite natural number $\kappa$ such that $|\rho^{-n} i_n| < 1$ for each $n < \kappa$. Therefore $|i_n| < \rho^n$ for each $n < \kappa$. If $n$ is infinite and $n < \kappa$, then $i_n \approx 0$ by the Criterion for Iotas. If $n \in N$, then $i_n \approx 0$ by assumption. This completes our proof. □

Here is another useful fact. Each decreasing sequence, over $N$, of positive megas is bounded below by a positive mega. This is analogous to Corollary 2.7.8.

7.12. LEMMA. *Let $(\kappa_n)$ be a decreasing sequence, over $N$, of positive megas. Then there is a positive mega $\Omega$ such that $\Omega < \kappa_n$ for each $n \in N$.*

*Proof.* Since ${}^*\mathbb{R}$ is sequentially comprehensive, there is an internal sequence $(s_n)$ such that $s_n = \kappa_n$ for each $n \in N$. Let $(u_n)$ be the internal sequence obtained from $(s_n)$ by replacing each of its zero terms (if any) by 1. Let $(t_n)$ be the internal sequence such that for each $n \in {}^*N$,

$$t_n = 1/\min\{|u_1|, ..., |u_n|\}.$$

For each $n \in N$, $t_n = 1/\kappa_n$; so $t_n \approx 0$. By the Fifth Principle of Permanence, there is an infinite natural number $\kappa$ such that $t_n \approx 0$ for each $n \leqslant \kappa$, $n \in {}^*N$. In particular, $t_\kappa \approx 0$; i.e.,

$$1/\min\{|u_1|, ..., |u_\kappa|\} \approx 0.$$

By the Criterion for Iotas there is an infinite natural number $\nu$ such that

$$1/\min\{|u_1|, ..., |u_\kappa|\} < \rho^\nu,$$

thus

$$\rho^{-\nu} < \min\{|u_1|, ..., |u_\kappa|\}.$$

Therefore $\rho^{-\nu} < |u_n|$ for each $n \leqslant \kappa$, $n \in {}^*N$; in particular, $\rho^{-\nu} < |u_n|$ for each $n \in N$. But $\Omega = \rho^{-\nu}$ is a positive mega. This completes our proof. □

(Of course, $u_n = \kappa_n$ for each $n \in N$.)

The set of natural numbers included in $M_0$ is of particular importance; in a sense, this set acts as the natural number set for ${}^{\rho}\mathbb{R}$. Let

$${}^{\rho}N = {}^*N \cap M_0;$$

so

$${}^{\rho}N = {}^*N - \{\kappa \mid \kappa \text{ is a mega}\}.$$

Our next principle of permanence is analogous to the First Principle of Permanence 2.7.2.

7.13. SIXTH PRINCIPLE OF PERMANENCE. *Let $A$ be an internal subset of ${}^*N$ such that ${}^{\rho}N \subset A$. Then there is a positive mega $\Omega$ such that $\kappa \in A$ for each $\kappa < \Omega$, $\kappa \in {}^*N$.*

*Proof.* Either $A = {}^*N$ or there is an infinite natural number which is not a member of $A$. In the former case there is nothing to prove. In the latter case ${}^*N - A$ is a nonempty, internal subset of ${}^*N$. Thus ${}^*N - A$ has a smallest member, which must be infinite and positive, say $\Omega$. Therefore, $\kappa \in A$ if $\kappa < \Omega$ and $\kappa \in {}^*N$. □

It is useful to formulate this principle of permanence as follows.

7.14. LEMMA. *Let $q \in {}^\rho N$, and let $A$ be an internal subset of ${}^*N$ such that*

$$\{m \in {}^\rho N \mid m > q\} \subset A.$$

*Then there is a positive mega $\Omega$ such that $\kappa \in A$ for each positive mega $\kappa$, $\kappa < \Omega$.*

*Proof.* Let

$$B = A \cup \{n \in {}^\rho N \mid n \leqslant q\}.$$

Notice that $B$ is an internal subset of ${}^*N$; moreover, ${}^\rho N \subset B$. By the Sixth Principle of Permanence, there is a positive mega $\Omega$ such that $\kappa \in B$ for each $\kappa < \Omega$, $\kappa \in {}^*N$. But no mega is a member of $\{n \in {}^\rho N \mid n \leqslant q\}$. Therefore $\kappa \in A$ for each positive mega $\kappa$, $\kappa < \Omega$.

CHAPTER 4

# FUNCTIONS IN $^\rho\mathcal{R}$

## 1. The function concept

Fundamentally, by a *function* in a field $\mathcal{F}$ we mean any map of a subset of $F$ into $F$. For the field $^\rho\mathcal{R}$, therefore, each map of a subset of $^\rho R$ into $^\rho R$ is a function in this field. For example, the identity map $\{(\gamma, \gamma) \mid \gamma \in {}^\rho R\}$ is a function in $^\rho\mathcal{R}$; any constant map, e.g. $\{(\gamma, 0) \mid \gamma \in {}^\rho R\}$, where 0 is the additive identity of $^\rho\mathcal{R}$, is a function in $^\rho\mathcal{R}$; the map $\{(\gamma, \gamma^2) \mid \gamma \in {}^\rho R\}$, where $\gamma^2 = \gamma \cdot \gamma$, is a function in $^\rho\mathcal{R}$.

Recall that each function in $\mathcal{R}$ (which we call a *standard* function) extends to a function in $^*\mathcal{R}$; indeed, the process by which $\mathcal{R}$ is extended to $^*\mathcal{R}$ automatically extends each standard function, say $f$, to a unique function in $^*\mathcal{R}$, called $^*f$.

There is a natural way of extending certain standard functions to functions in $^\rho\mathcal{R}$. Let $f$ be a standard function with domain $R$ such that $^*f$ has the property

$$(1.1) \qquad \forall xy\, [x \approx y \rightarrow {}^*f(x) \approx {}^*f(y)],$$

where the quantifiers refer to $M_0$. This property of $^*f$ allows us to associate a unique member of $^\rho R$ with each member of $^\rho R$. Namely, with each $\gamma \in {}^\rho R$ we associate $[{}^*f(x)]$ provided $x \in \gamma$. By assumption, $[{}^*f(y)] = [{}^*f(x)]$ if $y \in \gamma$. This defines a function in $^\rho\mathcal{R}$ which we denote by $^\rho f$ (read *the extension of $f$ to $^\rho\mathcal{R}$*). In short, for each $\gamma \in {}^\rho R$,

$$^\rho f(\gamma) = [{}^*f(x)],$$

where $x \in \gamma$. This definition is acceptable because $[{}^*f(x)] = [{}^*f(y)]$ for all $x, y \in \gamma$.

Here are some examples.

1.2. EXAMPLE. Let $f$ be the *squaring* function in $\mathcal{R}$; i.e.,

$$f = \{(t, t^2) \mid t \in R\}.$$

Then

$$^*f = \{(t, t^2) \mid t \in {}^*R\}.$$

First we must show that $^*f$ satisfies (1.1). Let $x, y \in M_0$, and let $x \approx y$; then $y = x + i$ for some iota $i$. Thus

$$^*f(y) - {}^*f(x) = (x + i)^2 - x^2 = 2xi + i^2 \approx 0$$

since $2xi \approx 0$ (by Lemma 3.7.3). Therefore we can extend $f$ to $^\rho f$, the function in $^\rho\mathcal{R}$ such that for each $\gamma \in {}^\rho R$,

$$^\rho f(\gamma) = [{}^*f(x)] = [x^2] = \gamma^2,$$

provided that $x \in \gamma$. Thus

$$^\rho f = \{(\gamma, \gamma^2) \mid \gamma \in {}^\rho R\}.$$

1.3. EXAMPLE. The standard function sin (i.e., $\{(t, \sin t) \mid t \in R\}$) extends to the function $^*$sin in $^*\mathcal{R}$. We shall show that $^*$sin satisfies (1.1). Let $x, y \in M_0$, and let $x \approx y$; then $y = x + i$ for some iota $i$. Thus

$$^*\sin(y) - {}^*\sin(x) = 2\,{}^*\sin(\tfrac{1}{2}i)\cdot{}^*\cos(x + \tfrac{1}{2}i) \approx 0$$

since $^*\sin(\frac{1}{2}i) \approx 0$ and $|{}^*\cos(x + \frac{1}{2}i)| \leqslant 1$. Therefore the standard function sin extends to the function $^\rho$sin defined as follows. For each $\gamma \in {}^\rho R$,

$$^\rho\sin(\gamma) = [{}^*\sin(x)],$$

where $x \in \gamma$.

Of course, some standard functions cannot be extended to $^\rho\mathcal{R}$ in this simple way, because (1.1) fails.

1.4. EXAMPLE. Let $f$ be the standard function such that for each $t \in R$,

$$f(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Then for each $t \in {}^*R$,

$$^*f(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

We shall show that $^*f$ does not satisfy (1.1). Let $i$ be any positive iota; then $i, -i \in 0$. But $^*f(i) = 1$ and $^*f(-i) = 0$. Therefore $^*f$ does not satisfy (1.1); so

$^{\rho}f$ does not exist. On the other hand,

$$\{(\gamma, 1) \mid \gamma \in {}^{\rho}R \text{ and } \gamma > 0\} \cup \{(\gamma, 0) \mid \gamma \in {}^{\rho}R \text{ and } \gamma \leqslant 0\}$$

is a function in $^{\rho}\mathbb{R}$ which we may regard, in a sense, as the extension of $f$ to $^{\rho}\mathbb{R}$. Our point is that this function is not yielded by our definition of $^{\rho}f$.

We shall say that $f$ *satisfies* (1.1) in case $^{*}f$ satisfies (1.1).

Here is a *sufficient* condition for (1.1).

1.5. LEMMA. *A standard function $f$ whose domain is an open interval $(a, b)$ satisfies* (1.1) *if $f$ meets a Lipschitz condition in each closed subinterval of $(a, b)$.*

*Proof.* $f$ meets a Lipschitz condition in $[a', b']$, a subinterval of $(a, b)$, provided that there is a standard number $k$ (which depends on $a'$ and $b'$) such that

$$(1.6) \qquad \forall xy\, [\,|f(y) - f(x)| \leqslant k\,|y - x|\,],$$

where the quantifiers refer to the closed interval $[a', b']$. Since (1.6) is true for $\mathbb{R}$, it is also true for $^{*}\mathbb{R}$ when interpreted in $^{*}\mathbb{R}$. Let $x, y \in {}^{*}(a, b)$, and let $x \approx y$; then for some $a', b' \in (a, b)$, $x, y \in {}^{*}[a', b']$. From (1.6),

$$|{}^{*}f(y) - {}^{*}f(x)| \leqslant k\,|y - x| \approx 0$$

since $k$ is standard and $y - x$ is an iota. So $^{*}f(x) \approx {}^{*}f(y)$. This completes our proof. □

We mention that the hypothesis of Lemma 1.5 is met by a standard function $f$ if $f'$ is continuous on the open interval $(a, b)$. In this case $f'$ is continuous on each closed subinterval $[a', b']$ of $(a, b)$; so $f'$ has a maximum value on $[a', b']$, which can serve as the $k$ of (1.6), in view of the Mean Value Theorem.

Next, we present a *necessary* condition for (1.1). First we establish the following fact.

1.7. LEMMA. *Let $f$ be a standard function, and let $a \in$ dom$f$. Then $f$ is continuous at $a$ if*

$$\forall x\, [x \approx a \rightarrow f(x) \approx f(a)].$$

*Proof.* Assume that $f$ is not continuous at $a$. Then there is a standard positive $\epsilon$ such that

$$(1.8) \qquad \forall \delta\, \exists x\, [\,|x - a| < \delta \wedge |f(x) - f(a)| \geqslant \epsilon\,]$$

is true for $\mathcal{R}$. Therefore, (1.8) is true for ${}^*\mathcal{R}$ when interpreted in ${}^*\mathcal{R}$. In particular, corresponding to each $n \in N$ there is an $x_n \in {}^*R$ such that

$$(1.9) \qquad |x_n - a| < \rho^n, \qquad |f(x_n) - f(a)| \geqslant \epsilon..$$

Since ${}^*\mathcal{R}$ is sequentially comprehensive, there is an internal sequence $(y_n)$, $n \in {}^*N$, such that $y_n = x_n$ for each $n \in N$. Now

$$A = \{n \in {}^*N \mid |y_n - a| < \rho^n \wedge |f(y_n) - f(a)| \geqslant \epsilon\}$$

is an internal subset of ${}^*N$; moreover, from (1.9), $N \subset A$. By the First Principle of Permanence 2.7.2, $A$ has an infinite member, say $\nu$. Thus

$$|y_\nu - a| < \rho^\nu, \qquad |f(y_\nu) - f(a)| \geqslant \epsilon.$$

Thus $y_\nu \approx a$, yet $f(y_\nu) \not\approx f(a)$ since $\epsilon$ is standard and positive. We conclude that ${}^*f$ does not satisfy (1.1). This establishes our lemma. □

1.10. COROLLARY. *Each standard function that satisfies* (1.1) *is continuous.*

We have established that our procedure for extending standard functions to ${}^\rho\mathcal{R}$ applies only to certain continuous functions. If a standard function $f$ is not continuous on its domain, then $f$ does not satisfy (1.1).

We emphasize that not every continuous function satisfies (1.1); here is an example.

1.11. EXAMPLE. *Let $f$ be the standard function such that for each $t \in (-1, 1)$,*

$$f(t) = \begin{cases} -1/\ln|t| & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Now it is well known that $f$ is continuous on $(-1, 1)$; we shall show that $f$ does not satisfy (1.1). Let $\kappa$ be an infinite natural number such that $\kappa\rho < 1$; so $\rho^\kappa$ is an iota. If $f$ satisfies (1.1), in particular $f(\rho^\kappa) \approx f(0)$, i.e., $f(\rho^\kappa) \approx 0$. If $f(\rho^\kappa)$ is an iota, so is $f(\rho^\kappa)/\rho^2$ (by our Criterion for Iotas). But

$$\frac{f(\rho^\kappa)}{\rho^2} = \frac{-1}{\rho^2 \ln \rho^\kappa} = \frac{-1}{\kappa\, \rho^2 \ln \rho} > \frac{-1}{\rho \ln \rho},$$

which is infinite. So $f(\rho^\kappa)/\rho^2$ is not an iota; thus $f(\rho^\kappa)$ is not an iota. In view of Lemma 1.5, and the comment following that lemma, we see that $0 \notin \operatorname{dom} f'$; this is easy to show directly. Continuing our example, let $t \in (-1, 1)$, where $t \neq 0$, and let $i$ be any iota; we shall prove that

$$(1.12) \qquad f(t + i) \approx f(t).$$

Now

$$f(t+i)-f(t)=\frac{-1}{\ln|t+i|}+\frac{1}{\ln|t|}$$
$$=\frac{\ln|t+i|-\ln|t|}{\ln|t+i|\ln|t|}$$
$$=\frac{i}{c\ln|t+i|\ln|t|}$$

by the Mean Value Theorem, where $c$ is between $t$ and $t+i$. Since $c$ is not an iota, $1/c$ is not a mega, so $i/c \approx 0$. Moreover, each of $\ln|t|$ and $\ln|t+i|$ is in $M_0 - M_1$ (since $t$ is a nonzero standard number). We conclude that $f(t+i) \approx f(t)$.

## 2. More functions

In Section 1 we showed how any standard function $f$ that satisfies (1.1), yields a unique function $^{\rho}f$ in $^{\rho}\mathcal{R}$. Here we point out that our construction applies equally well to any internal function in $^{*}\mathcal{R}$ that satisfies condition (1.1).

Let $f$ be an internal function in $^{*}\mathcal{R}$ (not necessarily rooted in a standard function) such that

(2.1) $\forall xy\,[x \approx y \rightarrow f(x) \approx f(y)]$,

where the quantifiers refer to $M_0 \cap \operatorname{dom} f$. This property of $f$ allows us to define a function in $^{\rho}\mathcal{R}$, denoted by $^{\rho}f$, as follows. Let $x \in M_0 \cap \operatorname{dom} f$, and let $\gamma = [x]$; then we say that $\gamma \in \operatorname{dom} {}^{\rho}f$ and define

(2.2) $^{\rho}f(\gamma) = [f(x)]$.

We now present some examples.

2.3. EXAMPLE. Let $f$ be the internal function in $^{*}\mathcal{R}$ such that for each $t \in {}^{*}R$,

$$f(t) = \sin \kappa t,$$

where $\kappa \in M_0$ (so $\kappa$ is not a mega). Take $x,y \in M_0$ so that $y - x = i \approx 0$; then

$$f(y)-f(x) = \sin\kappa y - \sin\kappa x$$
$$= 2\sin\tfrac{1}{2}\kappa(y-x)\cos\tfrac{1}{2}\kappa(y+x)$$
$$= 2\sin\tfrac{1}{2}\kappa i\cos\kappa(x+\tfrac{1}{2}i)$$
$$\approx 0$$

since $\sin\frac{1}{2}\kappa i \approx 0$ (note that $\frac{1}{2}\kappa i \approx 0$) and $|\cos\kappa(x+\frac{1}{2}i)| \leqslant 1$. Thus $f$ satisfies (2.1); so our procedure yields a function ${}^{\rho}f$ in ${}^{\rho}\mathcal{R}$, where ${}^{\rho}f(\gamma) = [\sin\kappa x]$, $x \in \gamma$. If $\kappa$ is a mega (i.e., $\kappa \notin M_0$), then (2.1) is *not* satisfied by $f$. We can choose $x$ and $y$ so that $y - x = 1/\kappa$, therefore $\sin\frac{1}{2}\kappa i = \sin\frac{1}{2} \not\approx 0$ and $\cos\frac{1}{2}\kappa(y+x) = \cos(\kappa x + \frac{1}{2})$ is not an iota for appropriate $x$ (e.g., $x = 0$).

2.4. EXAMPLE. Let $\kappa \in M_0$, $\kappa > 0$. Then $\delta$ is a Dirac delta function, where

$$\delta(t) = \sqrt{\kappa/\pi}\,\exp(-\kappa t^2)$$

for each $t \in {}^*R$. Notice that for each $t \in {}^*R$,

$$\delta'(t) = -2\kappa t\sqrt{\kappa/\pi}\,\exp(-\kappa t^2) = -2\kappa t\,\delta(t).$$

Take $x, y \in M_0$, where $y - x = i \approx 0$. By the Mean Value Theorem, for some $c$ between $x$ and $y$

$$\delta(y) - \delta(x) = (y-x)\,\delta'(c) = -2\kappa c i\,\delta(c).$$

Clearly $2\kappa c i \approx 0$. If $c \not\approx 0$, then $\delta(c) \simeq 0$; so $2\kappa c i\,\delta(c) \approx 0$. If $c \simeq 0$, then $0 < \exp(-\kappa c^2) \leqslant 1$, and it follows that $\delta(c) \in M_0$; so $-2\kappa c i\,\delta(c) \approx 0$. We conclude that $\delta(y) - \delta(x) \approx 0$; so $\delta$ satisfies (2.1). This means that $\delta$ yields a unique function in ${}^{\rho}\mathcal{R}$, denoted by ${}^{\rho}\delta$, where ${}^{\rho}\delta(\gamma) = [\delta(x)]$ for each $\gamma \in {}^{\rho}R$, where $x \in \gamma$.

Our next example shows that for a function in ${}^{\rho}\mathcal{R}$, say ${}^{\rho}f$, the function $f$ involved is not necessarily unique.

2.5. EXAMPLE. Let $i$ be any positive iota, and let $f$ be the internal function such that

$$f(t) = \begin{cases} i & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Clearly $f$ satisfies (2.1); so $f$ yields ${}^{\rho}f$, where for each $\gamma \in {}^{\rho}R$ and $x \in \gamma$, ${}^{\rho}f(\gamma) = [f(x)]$; thus

$${}^{\rho}f(\gamma) = \begin{cases} [i] & \text{if } \gamma > 0, \\ 0 & \text{if } \gamma \leqslant 0. \end{cases}$$

But $[i] = 0$, so ${}^{\rho}f(\gamma) = 0$ for each $\gamma \in {}^{\rho}R$. Of course, the internal function $\{(t, 0) \mid t \in {}^*R\}$ also satisfies (2.1), and yields $\{(\gamma, 0) \mid \gamma \in {}^{\rho}R\}$ as well.

Here is an internal function to which our procedure for obtaining a function in ${}^{\rho}\mathbb{R}$ does not apply.

2.6. EXAMPLE. Let $f$ be the internal function such that

$$f(t) = \begin{cases} \rho & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Let $i$ be any positive iota; then $f(i) \not\approx f(0)$, so $f$ does *not* satisfy (2.1). Therefore, $f$ does not yield a function in ${}^{\rho}\mathbb{R}$.

The following lemmas help us decide whether a given internal function satisfies (2.1).

2.7. LEMMA. *Let $f$ and $g$ be internal functions that satisfy* (2.1). *Then $f+g$ and $f-g$ satisfy* (2.1). *Moreover, $f \cdot g$ satisfies* (2.1) *provided that $f(x), g(x) \in M_0$ whenever $x \in M_0$.*

*Proof.* We are given that

$$\forall xy\ [x \approx y \to (f(x) \approx f(y) \wedge g(x) \approx g(y))],$$

where the quantifiers refer to $M_0 \cap \operatorname{dom} f \cap \operatorname{dom} g$. So

$$f(x) - f(y) \approx 0, \qquad g(x) - g(y) \approx 0.$$

The sum, or difference, of two iotas is an iota; thus

$$[f(x) - f(y)] \pm [g(x) - g(y)] \approx 0,$$

i.e.,

$$f(x) \pm g(x) \approx f(y) \pm g(y).$$

So both $f+g$ and $f-g$ satisfy (2.1). Next, suppose as well that $f(x), g(x) \in M_0$ whenever $x \in M_0$. Then

$$f(x)g(x) - f(y)g(y) = f(x)[g(x) - g(y)] + g(y)[f(x) - f(y)]$$
$$\approx 0$$

since the product of an iota and a member of $M_0$ is an iota. Thus $f(x)g(x) \approx f(y)g(y)$; so $f \cdot g$ satisfies (2.1). □

2.8. LEMMA. *Let $f$ and $g$ be internal functions that satisfy* (2.1) *such that no value of $g$ is an iota and $f(x) \in M_0$ if $x \in M_0$. Then $f/g$ satisfies* (2.1).

*Proof.* Let $h = 1/g$; then $h$ satisfies (2.1) since

$$h(x) - h(y) = \frac{1}{g(x)} - \frac{1}{g(y)} = \frac{g(y) - g(x)}{g(x)\,g(y)} \approx 0$$

if $g(x) \approx g(y)$. Moreover, $h(x) \in M_0$ if $x \in M_0$; thus, by Lemma 2.7, $f \cdot h$ satisfies (2.1), i.e., $f/g$ satisfies (2.1). □

2.9. LEMMA. *Let $f$ and $g$ be internal functions that satisfy* (2.1) *and such that $g(x) \in M_0$ if $x \in M_0$. Then the composite function $f \circ g$ satisfies* (2.1).

*Proof.* Let $x \approx y$, where $x, y \in M_0 \cap \operatorname{dom} f \circ g$. Then $g(x) \approx g(y)$; thus $f(g(x)) \approx f(g(y))$, i.e.,

$$[f \circ g](x) \approx [f \circ g](y).$$

## 3. Continuity

We shall base our concept of *continuity* in the nonarchimedean field ${}^{\rho}\mathcal{R}$, on the metric induced by the nonarchimedean valuation $v$ defined in Section 3.3 (for the induced metric, see Section 3.4). The idea is that in ${}^{\rho}\mathcal{R}$, a function $f$ is continuous at $\gamma$, where $\gamma \in \operatorname{dom} f$, if each sequence $(f(\gamma_n))$ converges to $f(\gamma)$, where $(\gamma_n)$ is a sequence such that:

(1) $(\gamma_n)$ converges to $\gamma$;

(2) $\gamma_n \in \operatorname{dom} f$ for each $n \in N$.

This sort of continuity is called *sequential* continuity (for short, seq-continuity).

3.1. DEFINITION. A function $f$ is *sequentially continuous at* $\gamma$, $\gamma \in \operatorname{dom} f$, provided that $\lim(f(\gamma_n)) = f(\gamma)$ for each sequence $(\gamma_n)$ such that:

(i) $\lim(\gamma_n) = \gamma$;

(ii) $\gamma_n \in \operatorname{dom} f$ for each $n \in N$.

Before illustrating sequential continuity, recall that for each sequence $(\alpha_n)$, and for each $\alpha \in {}^{\rho}R$,

$$\lim(\alpha_n) = \alpha \quad \text{iff} \quad \lim(d(\alpha_n, \alpha)) = 0$$

$$\text{iff} \quad \lim(|\alpha_n - \alpha|_v) = 0,$$

where $|\gamma|_v = e^{-v(\gamma)}$ for each $\gamma \in {}^{\rho}R$, and $v([x]) = {}^{0}(\log_{\rho}|x|)$. For example,

$$|[\rho]|_v = e^{-v([\rho])} = 1/e.$$

Recall Theorem 3.5.13. If $\lim(\alpha_n) = \alpha$, where $\alpha \neq 0$, then

(3.2) $$\exists j \,\forall m \,[m > j \to |\alpha_m|_v = |\alpha|_v], \quad j, m \in N.$$

We shall use these facts in our examples of sequential continuity, which follow.

3.3. EXAMPLE. Let $f$ be the function such that $f(\gamma) = 1/\gamma$ for each $\gamma \neq 0$. We shall show that $f$ is seq-continuous at $[\rho]$. Let $(\gamma_n)$ be any sequence such that $\lim(\gamma_n) = [\rho]$ and $\gamma_n \neq 0$ for each $n \in N$. We must show that $\lim(1/\gamma_n) = [1/\rho]$; i.e.,

(3.4) $$\forall \epsilon \,\exists q \,\forall m \,[m > q \to |1/\gamma_m - [1/\rho]|_v < \epsilon], \quad \epsilon \in R, \epsilon > 0, q, m \in N.$$

But

$$\left|\frac{1}{\gamma_m} - \frac{1}{[\rho]}\right|_v = \left|\frac{[\rho] - \gamma_m}{[\rho]\,\gamma_m}\right|_v \qquad \text{since } {}^{\rho}\mathcal{R} \text{ is a field}$$

$$= \frac{|\gamma_m - [\rho]|_v}{|[\rho]|_v\,|\gamma_m|_v} \qquad \text{by properties of } |\ \ |_v \text{ (see Section 1.5).}$$

Now $|[\rho]|_v = 1/\mathrm{e}$. Also, $\lim(\gamma_n) = [\rho]$; so by (3.2),

$$\exists j \,\forall m \,[m > j \to |\gamma_m|_v = 1/\mathrm{e}], \quad j, m \in N.$$

Therefore, for $m > j$,

(3.5) $$|1/\gamma_m - [1/\rho]|_v = \mathrm{e}^2\,|\gamma_m - [\rho]|_v.$$

But $\lim(\gamma_n) = [\rho]$, so $\lim(|\gamma_n - [\rho]|_v) = 0$. We conclude from (3.5) that (3.4) is correct, i.e., $\lim(1/\gamma_n) = [1/\rho]$. This proves that $f$ is sequentially continuous at $[\rho]$.

Continuing this example, let us show that $f$ is seq-continuous at $\gamma$ provided that $\gamma \neq 0$. Let $(\gamma_n)$ be any sequence such that $\lim(\gamma_n) = \gamma$ and $\gamma_n \neq 0$ for each $n \in N$. By (3.2) there is a standard natural number $j$ such that $|\gamma_m|_v = |\gamma|_v$ for each $m > j$. We shall show that $\lim(1/\gamma_n) = 1/\gamma$. Now, for $m > j$

(3.6) $$\left|\frac{1}{\gamma_m} - \frac{1}{\gamma}\right|_v = \frac{|\gamma_m - \gamma|_v}{|\gamma|_v\,|\gamma_m|_v} = \frac{|\gamma_m - \gamma|_v}{|\gamma|_v\,|\gamma|_v}.$$

Since $\lim(|\gamma_n - \gamma|_v) = 0$, we conclude from (3.6) that $\lim(|1/\gamma_n - 1/\gamma|_v) = 0$. Thus $\lim(1/\gamma_n) = 1/\gamma$. This proves that $f$ is seq-continuous at $\gamma$; so $f$ is seq-continuous at each member of its domain.

3.7. EXAMPLE. Let $f$ be the function such that $f(\gamma) = \gamma^2$ for each $\gamma \in {}^\rho R$. We shall show that $f$ is seq-continuous at each member of its domain. Let $\alpha \in {}^\rho R$, and let $(\alpha_n)$ be any sequence such that $\lim(\alpha_n) = \alpha$. We wish to show that $\lim(\alpha_n^2) = \alpha^2$. First assume that $\alpha \neq [0]$. From (3.2), $|\alpha_m|_v = |\alpha|_v$ if $m$ is sufficiently large; so

$$|\alpha_m + \alpha|_v \leqslant \max\{|\alpha_m|_v, |\alpha|_v\} = |\alpha|_v.$$

Thus

$$\begin{aligned} |\alpha_m^2 - \alpha^2|_v &= |(\alpha_m - \alpha)(\alpha_m + \alpha)|_v \quad &&(\text{since } {}^\rho\mathcal{R} \text{ is a field}) \\ &= |\alpha_m - \alpha|_v\, |\alpha_m + \alpha|_v \quad &&(\text{by (4) of Section 1.5}) \\ &\leqslant |\alpha|_v\, |\alpha_m - \alpha|_v. \end{aligned}$$

By assumption, $\lim(|\alpha_n - \alpha|_v) = 0$; we conclude that $\lim(|\alpha_n^2 - \alpha^2|_v) = 0$, thus $\lim(\alpha_n^2) = \alpha^2$. This proves that $f$ is seq-continuous at $\alpha$. Next, assume that $\alpha = [0]$. Then $\lim(\alpha_n) = [0]$; so $\lim(|\alpha_n|_v) = 0$. But $|\alpha_n^2|_v = |\alpha_n|_v\, |\alpha_n|_v$; so

$$\lim(|\alpha_n^2|_v) = (\lim(|\alpha_n|_v))^2 = 0.$$

Thus $f$ is seq-continuous at $[0]$.

3.8. EXAMPLE. Let $f$ be the function such that

$$f(\gamma) = \begin{cases} [1] & \text{if } \gamma \geqslant 0, \\ [0] & \text{if } \gamma < 0. \end{cases}$$

We shall show that $f$ is not seq-continuous at $[0]$. Notice that

$$\lim(|[-\rho]^n|_v) = \lim(e^{-n}) = 0.$$

The corresponding sequence of values of $f$ is

$$(f([-\rho]^n)) = ([0], [1], [0], [1], \ldots),$$

which does not converge to $f([0]) = [1]$; indeed, this sequence does not converge. We conclude that $f$ is not seq-continuous at $[0]$. Continuing this example, let us show that $f$ is seq-continuous at any other member of ${}^\rho R$, say $\gamma$. Take $\gamma > [0]$ and let $(\gamma_n)$ be a sequence such that $\lim(\gamma_n) = \gamma$. We claim that

$$(3.9) \qquad \exists q\, \forall m\, [m > q \to \gamma_m > [0]], \quad q, m \in N.$$

If not, corresponding to each standard natural number $q$ there is a standard natural number $m$, $m > q$, such that $\gamma_m \leqslant [0]$. So $|\gamma_m - \gamma| \geqslant \gamma$, thus

$|\gamma_m - \gamma|_v \geqslant |\gamma|_v$ by Lemma 3.4.5. But $|\gamma|_v$ is standard and positive; therefore $\lim(|\gamma_n - \gamma|_v) \neq 0$, so $\lim(\gamma_n) \neq \gamma$. This contradiction verifies (3.9). Thus, for $m > q$, $f(\gamma_m) = [1] = f(\gamma)$. A similar argument works for $\gamma < [0]$. We conclude that $f$ is seq-continuous at $\gamma$ provided that $\gamma \neq [0]$.

## 4. S-continuity

In view of Lemma 2.10.9, the notion of S-continuity in ${}^*\mathcal{R}$ is meaningful in the context of ${}^{\rho}\mathcal{R}$ provided that we interpret absolute value as $|\ |_v$.

Let us be quite precise.

4.1. DEFINITION OF S-CONTINUITY. A function $f$ is *S-continuous at* $\alpha$, $\alpha \in \operatorname{dom} f$, if

$$\forall\epsilon\, \exists\delta\, \forall\gamma\, [|\gamma - \alpha|_v < \delta \to |f(\gamma) - f(\alpha)|_v < \epsilon], \tag{4.2}$$

where the first two quantifiers refer to positive standard numbers, and the last quantifier refers to $\operatorname{dom} f$.

We shall show that each function $f$ is S-continuous at $\alpha$ iff $f$ is seq-continuous at $\alpha$. First we present some examples.

4.3. EXAMPLE. The *absolute value* function $|\ |$ is S-continuous at each $\alpha \in {}^{\rho}R$. Here

$$|\gamma| = \begin{cases} \gamma & \text{if } \gamma \geqslant 0, \\ -\gamma & \text{if } \gamma < 0. \end{cases}$$

In fact, we claim that

$$\forall\epsilon\, \forall\gamma\, [|\gamma - \alpha|_v < \epsilon \to ||\gamma| - |\alpha||_v < \epsilon]. \tag{4.4}$$

Notice that $|\gamma + \alpha| < |\gamma - \alpha|$ if $\alpha$ and $\gamma$ have opposite signs; in this case,

$$||\gamma| - |\alpha|| = |\gamma + \alpha| < |\gamma - \alpha|.$$

Thus, by Lemma 3.4.5,

$$||\gamma| - |\alpha||_v \leqslant |\gamma - \alpha|_v < \epsilon$$

by assumption. If $\alpha$ and $\gamma$ have the same sign, then

$$||\gamma| - |\alpha|| = |\gamma - \alpha|$$

so by (2), Section 1.5,

$$||\gamma| - |\alpha||_v = |\gamma - \alpha|_v < \epsilon$$

by assumption. This verfies (4.4). So | | is S-continuous at each member of ${}^{\rho}R$.

4.5. EXAMPLE. Let $f$ be the function such that for each $\gamma \in {}^{\rho}R$,

$$f(\gamma) = \begin{cases} [\rho] & \text{if } \gamma > [0], \\ [0] & \text{if } \gamma \leqslant [0]. \end{cases}$$

We claim that $f$ is not S-continuous at [0]. To see this, in (4.2) take $\epsilon = 1/(2e)$. Corresponding to each positive standard $\delta$, there is a standard natural number $m$ such that

$$|[\rho]^m|_v = e^{-m} < \delta$$

(since $\lim(e^{-n}) = 0$). For each choice of $\delta$, in (4.2), take $\gamma = [\rho]^m$. Then $|\gamma|_v < \delta$, but

$$|f(\gamma)|_v = |[\rho]|_v = 1/e > \epsilon.$$

So (4.2) is not satisfied; i.e., $f$ is not S-continuous at [0].

Next we shall show that $f$ is S-continuous at $\alpha$ if $\alpha \neq [0]$. Take $\alpha > [0]$. Choose $\epsilon > 0$ in (4.2), take $\delta = |\alpha|_v$, and let $\gamma$ be such that $|\gamma - \alpha|_v < \delta$. By Lemma 3.4.7, $\gamma > [0]$. So

$$|f(\gamma) - f(\alpha)|_v = |[\rho] - [\rho]|_v = |[0]|_v = 0 < \epsilon.$$

Thus (4.2) is satisfied; i.e., $f$ is S-continuous at $\alpha$. Finally take $\alpha < [0]$. Choose $\epsilon > 0$ in (4.2), take $\delta = |\alpha|_v$, and let $\gamma$ be such that $|\gamma - \alpha|_v < \delta$. By the Corollary 3.4.8, $\gamma < [0]$. So

$$|f(\gamma) - f(\alpha)|_v = |[0] - [0]|_v = |[0]|_v = 0 < \epsilon.$$

Thus (4.2) is satisfied; i.e., $f$ is S-continuous at $\alpha$.

We shall now establish the connection between S-continuity and seq-continuity.

4.6. THEOREM. *Let $f$ be any function in ${}^{\rho}\mathcal{R}$, and let $\alpha \in$ dom$f$. Then $f$ is S-continuous at $\alpha$ iff $f$ is seq-continuous at $\alpha$.*

*Proof.* (i) Assume that $f$ is S-continuous at $\alpha$. Let $(\alpha_n)$ be any sequence such that $\lim(\alpha_n) = \alpha$ and $\alpha_n \in \operatorname{dom} f$ for each $n \in N$. Then $\lim(|\alpha_n - \alpha|_v) = 0$, so

$$(4.7) \qquad \forall \epsilon\, \exists q\, \forall m\, [m > q \to |\alpha_m - \alpha|_v < \epsilon],$$

where the first quantifier refers to positive standard numbers, and the other quantifiers refer to standard natural numbers. We shall show that $\lim(f(\alpha_n)) = f(\alpha)$, i.e. $\lim(|f(\alpha_n) - f(\alpha)|_v) = 0$, namely

$$(4.8) \qquad \forall \epsilon\, \exists q\, \forall m\, [m > q \to |f(\alpha_m) - f(\alpha)|_v < \epsilon],$$

where the quantifiers have the same designation as in (4.7). By assumption, $f$ is S-continuous at $\alpha$, i.e.,

$$(4.9) \qquad \forall \epsilon\, \exists \delta\, \forall \gamma\, [|\gamma - \alpha|_v < \delta \to |f(\gamma) - f(\alpha)|_v < \epsilon].$$

Fix $\epsilon$ positive and standard. From (4.9), there is a positive standard $\delta$ such that

$$(4.10) \qquad \forall \gamma\, [|\gamma - \alpha|_v < \delta \to |f(\gamma) - f(\alpha)|_v < \epsilon].$$

From (4.7) with $\epsilon = \delta$, there is a standard natural number $q$ such that

$$(4.11) \qquad \forall m\, [m > q \to |\alpha_m - \alpha|_v < \delta].$$

So, from (4.11) and (4.10),

$$\forall m\, [m > q \to |f(\alpha_m) - f(\alpha)|_v < \epsilon].$$

This establishes (4.8), i.e., $\lim(f(\alpha_n)) = f(\alpha)$; so $f$ is seq-continuous at $\alpha$.

(ii) Assume that $f$ is not S-continuous at $\alpha$. Then there is a positive standard number $\epsilon$ such that

$$(4.12) \qquad \forall \delta\, \exists \gamma\, [|\gamma - \alpha|_v < \delta \wedge |f(\gamma) - f(\alpha)|_v \geqslant \epsilon].$$

We shall show that $f$ is not seq-continuous at $\alpha$. In (4.12) take $\delta = \frac{1}{2}$; then for some $\gamma_1 \in \operatorname{dom} f$,

$$|\gamma_1 - \alpha|_v < \tfrac{1}{2}, \qquad |f(\gamma_1) - f(\alpha)|_v \geqslant \epsilon.$$

In (4.12) take $\delta = (\frac{1}{2})^2$; then for some $\gamma_2 \in \operatorname{dom} f$,

$$|\gamma_2 - \alpha|_v < (\tfrac{1}{2})^2, \qquad |f(\gamma_2) - f(\alpha)|_v \geqslant \epsilon.$$

In (4.12) take $\delta = (\frac{1}{2})^3$; then for some $\gamma_3 \in \operatorname{dom} f$,

$$|\gamma_3 - \alpha|_v < (\tfrac{1}{2})^3, \qquad |f(\gamma_3) - f(\alpha)|_v \geqslant \epsilon.$$

More generally, take $\delta = (\frac{1}{2})^n$, where $n \in N$; then for some $\gamma_n \in \operatorname{dom} f$,

$$|\gamma_n - \alpha|_v < (\tfrac{1}{2})^n, \qquad |f(\gamma_n) - f(\alpha)|_v \geqslant \epsilon.$$

In this way, (4.12) yields a sequence $(\gamma_n)$ such that $\lim(\gamma_n) = \alpha$, $\gamma_n \in \operatorname{dom} f$ for each $n \in N$, and $|f(\gamma_n) - f(\alpha)|_v \geqslant \epsilon$ for each $n \in N$. Thus $(f(\gamma_n))$ does not converge to $f(\alpha)$. We conclude that $f$ is not seq-continuous at $\alpha$. This completes our proof of the theorem. □

What about Q-continuity in $^\rho\mathcal{R}$? Well, the obvious way of formulating Q-continuity in $^\rho\mathcal{R}$ yields a trivial concept. Let us demonstrate this point.

4.13. DEFINITION OF Q-CONTINUITY. A function $f$ is Q-*continuous at* $\alpha$, $\alpha \in \operatorname{dom} f$, if

$$\forall\epsilon\, \exists\delta\, \forall\gamma\, [|\gamma - \alpha|_v < \delta \to |f(\gamma) - f(\alpha)|_v < \epsilon], \tag{4.14}$$

where the first two quantifiers refer to positive real numbers (i.e., members of $^*R$) and the last quantifier refers to $\operatorname{dom} f$.

Unfortunately, each function $f$ in $^\rho\mathcal{R}$ for which $\alpha \in \operatorname{dom} f$, satisfies (4.14). To see this, fix $\epsilon \in {}^*R$, $\epsilon > 0$, and choose $\delta$ so that $\delta > 0$ and $\delta \simeq 0$ (e.g., take $\delta = \rho$). Notice that for each $\gamma \in {}^\rho R$,

$$|\gamma - \alpha|_v < \delta \leftrightarrow \gamma = \alpha$$

(since $|\beta|_v$ is standard and positive iff $\beta \neq [0]$). But if $\gamma = \alpha$, then

$$|f(\gamma) - f(\alpha)|_v = |[0]|_v = 0 < \epsilon,$$

so (4.14) is satisfied. Thus $f$ is Q-continuous at $\alpha$ provided that $\alpha \in \operatorname{dom} f$.

For this reason, we shall concentrate on S-continuity in $^\rho\mathcal{R}$, i.e., on seq-continuity.

## 5. Functions $^\rho f$ and continuity

Let $f$ be any standard function which satisfies (1.1); so $f$ yields $^\rho f$, a function in $^\rho\mathcal{R}$. In this section we shall discuss the continuity of $^\rho f$.

First, we introduce some ideas and terminology.

5.1. DEFINITION. Let $\alpha, \beta \in {}^\rho R$; we say that $\alpha$ is *infinitely close to* $\beta$ (in symbols, $\alpha \simeq \beta$) if $a \simeq b$ for each $a \in \alpha$ and $b \in \beta$.

Since any two members of $\alpha$ differ by an iota, and any two members of $\beta$ differ by an iota, it is clear that

$$\alpha \simeq \beta \quad \text{iff} \quad \exists ab\, [a \in \alpha \wedge b \in \beta \wedge a \simeq b].$$

Here is a fact about convergence in ${}^{\rho}\mathcal{R}$.

5.2. LEMMA. *Let* $\lim(\gamma_n) = \gamma$. *Then*

$$\exists q \forall m \, [m > q \to \gamma_m \simeq \gamma].$$

*Proof.* Assume that $\lim(\gamma_n) = \gamma$ but

$$\forall q \exists m \, [m > q \wedge \gamma_m \not\simeq \gamma].$$

Then $\gamma_m - \gamma$ contains no infinitesimals; so $x \in \gamma_m - \gamma$ iff there is a positive standard number $h$ such that $|x| > h$. Thus

$$|\gamma_m - \gamma|_v = \exp\{-{}^0(\log_\rho |x|)\} \geqslant \exp\{-{}^0(\log_\rho h)\}.$$

But $h$ is positive and standard, so $\log_\rho h \simeq 0$; thus ${}^0(\log_\rho h) = 0$. So $|\gamma_m - \gamma|_v \geqslant 1$. This proves that

$$\forall q \exists m \, [m > q \wedge |\gamma_m - \gamma|_v \geqslant 1];$$

therefore $\lim(|\gamma_n - \gamma|_v) \neq 0$. This contradiction establishes the lemma. □

Given an open interval $(a, b)$ in $\mathcal{R}$, we introduce ${}^{\rho}(a, b)$ a subset of ${}^{\rho}R$ defined as follows.

5.3. DEFINITION OF ${}^{\rho}(a, b)$. For each $\gamma \in {}^{\rho}R$,

$$\gamma \in {}^{\rho}(a, b) \quad \text{iff} \quad \exists t \, [t \in R \wedge a < t < b \wedge \gamma \simeq [t]]. \tag{5.4}$$

Remember that we identify $[t]$ with $t$ in case $t \in R$; so we sometimes write (5.4) more simply as

$$\gamma \in {}^{\rho}(a, b) \quad \text{iff} \quad \exists t \, [t \in R \wedge a < t < b \wedge \gamma \simeq t].$$

Similarly, we can regard the open interval

$$(a, b) = \{t \in R \mid a < t < b\}$$

as a subset of ${}^{\rho}R$, namely

$$\{[t] \mid t \in R \text{ and } a < t < b\}.$$

In this sense, ${}^{\rho}(a, b)$ is a superset of $(a, b)$. Moreover, if $t \in (a, b)$, so $t \in R$, then $[t + \epsilon] \in {}^{\rho}(a, b)$ for each infinitesimal $\epsilon$.

Here is a simple way of characterizing ${}^{\rho}(a, b)$.

5.5. LEMMA. *Let* $(a, b)$ *be an open interval in* $\mathcal{R}$. *Then* $\gamma \in {}^{\rho}(a, b)$ *iff* $\gamma \subset {}^{*}(a, b)$.

*Proof.* (i) Assume that $\gamma \in {}^{\rho}(a, b)$. Then $\gamma \simeq [t]$ for some $t \in R$ such that $a < t < b$. Thus $\gamma = [x]$, where $x \simeq t$ and $x \in {}^*R$. So $\gamma \subset {}^*(a, b)$.

(ii) Assume that $\gamma \subset {}^*(a, b)$. Let $\gamma = [x]$, where $x \in {}^*R$. By the Fundamental Theorem about Finite Numbers 2.1.7, there is a standard number $t$ such that $x \simeq t$. Since $a < x < b$ and $a, b \in R$, we see that $a < t < b$. By construction, $\gamma \simeq [t]$; so $\gamma \in {}^{\rho}(a, b)$. This completes our proof of the lemma. □

Notice that the converse of Lemma 5.2 is false. For example, consider the sequence $(\gamma_n)$, where $\gamma_n = \gamma + [\rho]$ for each $n \in N$. Here $\forall n\ [\gamma_n \simeq \gamma]$, yet $\lim(\gamma_n) \neq \gamma$.

We are now ready for a continuity result.

5.6. THEOREM. *Let $f$ be a standard function, whose domain includes $(a, b)$, such that $f'$ is continuous on $(a, b)$. Then $^{\rho}f$ is seq-continuous on $^{\rho}(a, b)$.*

*Proof.* In view of the comment that follows Lemma 1.5, $f$ satisfies (1.1); so the domain of $^{\rho}f$ includes $^{\rho}(a, b)$. To show that $^{\rho}f$ is seq-continuous on $^{\rho}(a, b)$, let $\gamma \in {}^{\rho}(a, b)$, and let $(\gamma_n)$ be a sequence such that $\lim(\gamma_n) = \gamma$, where $\gamma_n \in \text{dom}\,{}^{\rho}f$ for each $n \in N$. By Lemma 5.2, there is a standard natural number $q$ such that $\gamma_m \simeq \gamma$ for each $m > q$. Let $x_n \in \gamma_n$ for each $n \in N$, and let $x \in \gamma$. Then $x_m \simeq x$ for each $m > q$. Since $\gamma \in {}^{\rho}(a, b)$, there is a standard number $t$ such that $a < t < b$ and $\gamma \simeq t$. Thus $x \simeq t$ and $x_m \simeq t$ for each $m > q$. It follows that there are standard numbers $a'$ and $b'$ such that $a < a' < t < b' < b$ and $x_m \in [a', b']$ for each $m > q$. By assumption, $f'$ is continuous on the closed interval $[a', b']$, so $|f'|$ has a maximum value on $[a', b']$, say $B$ ($B \in R$); i.e., $|f'(s)| \leqslant B$ for each $s \in [a', b']$ ($s \in R$). Thus $|f'(t)| \leqslant B$ for each $t \in {}^*[a', b']$ ($t \in {}^*R$). By the Mean Value Theorem for $^*\mathcal{R}$, where $m > q$,

$$|f(x_m) - f(x)| = |x_m - x|\,|f'(t_m)| \leqslant B\,|x_m - x|$$

for some $t_m$ between $x_m$ and $x$. Thus

$$|{}^{\rho}f(\gamma_m) - {}^{\rho}f(\gamma)| \leqslant B\,|\gamma_m - \gamma|.$$

By Lemma 3.4.5,

$$|{}^{\rho}f(\gamma_m) - {}^{\rho}f(\gamma)|_v \leqslant |\gamma_m - \gamma|_v \quad \text{if } m > q, \tag{5.7}$$

since $|[B]|_v = 1$ if $B > 0$ (if $B = 0$, then $|[B]|_v = 0$). By assumption, $\lim(|\gamma_n - \gamma|_v) = 0$; it follows from (5.7) that

$$\lim(|{}^{\rho}f(\gamma_n) - {}^{\rho}f(\gamma)|_v) = 0,$$

so $\lim({}^{\rho}f(\gamma_n)) = {}^{\rho}f(\gamma)$. So $^{\rho}f$ is seq-continuous at $\gamma$. We conclude that $^{\rho}f$ is seq-continuous on $^{\rho}(a, b)$. □

We now turn to internal functions.

5.8. THEOREM. *Let $f$ be an internal function such that $f'$ is continuous on an open interval $^*(a, b)$ with standard endpoints $a$ and $b$, and such that the range of $f'$ is a subset of $M_0$. Then $f$ is seq-continuous on $^{\rho}(a, b)$.*

*Proof.* First we show that $f$ satisfies (2.1) on $^*(a, b)$. Let $x \approx y$, where $x, y \in {}^*(a, b)$; then

$$|f(x) - f(y)| = |x - y|\,|f'(t)| \leqslant B\,|x - y|,$$

where $B$ is the maximum value of $|f'|$ on the closed interval with endpoints $x$ and $y$ (i.e., $B = |f'(s)|$ for some $s$ between $x$ and $y$). But $B\,|x - y| \approx 0$ since $x - y \approx 0$ and $B \in M_0$ (see Lemma 3.7.3). This proves that the domain of $^{\rho}f$ includes $^{\rho}(a, b)$. To show that $^{\rho}f$ is seq-continuous on $^{\rho}(a, b)$, let $\gamma \in {}^{\rho}(a, b)$, and let $(\gamma_n)$ be a sequence such that $\lim(\gamma_n) = \gamma$, where $\gamma_n \in \operatorname{dom} {}^{\rho}f$ for each $n \in N$. By Lemma 5.2, there is a standard natural number $q$ such that $\gamma_m \simeq \gamma$ for each $m > q$. Let $x_n \in \gamma_n$ for each $n \in N$, and let $x \in \gamma$. Then $x_m \simeq x$ for each $m > q$. There are standard numbers $a'$ and $b'$ such that $a < a' < x < b' < b$ and $x_m \in [a', b']$ for each $m > q$. Since $f'$ is continuous on $^*[a', b']$, it follows that $|f'|$ has a maximum value on $^*[a', b']$, say $C$. By the Mean Value Theorem for $^*\mathcal{R}$,

$$|f(x_m) - f(x)| = |x_m - x|\,|f'(t_m)|$$

$$\leqslant C\,|x_m - x|$$

for some $t_m$ between $x_m$ and $x$. Thus

$$|{}^{\rho}f(\gamma_m) - {}^{\rho}f(\gamma)| \leqslant C\,|\gamma_m - \gamma|.$$

So, by Lemma 3.4.5,

$$|{}^{\rho}f(\gamma_m) - {}^{\rho}f(\gamma)|_v \leqslant |[C]|_v\,|\gamma_m - \gamma|_v. \tag{5.9}$$

But $|[C]|_v \leqslant e^j$ for some $j \in N$, since $C \in M_0$. Moreover, $\lim(|\gamma_n - \gamma|_v) = 0$ by assumption; so, from (5.9),

$$\lim(|{}^{\rho}f(\gamma_n) - {}^{\rho}f(\gamma)|_v) = 0,$$

i.e., $\lim({}^{\rho}f(\gamma_n)) = {}^{\rho}f(\gamma)$. Thus $^{\rho}f$ is seq-continuous at $\gamma$. We conclude that $^{\rho}f$ is seq-continuous on $^{\rho}(a, b)$. This completes our proof of the theorem. □

## 6. Differentiation

Let $f$ be any function in ${}^\rho\mathcal{R}$. The derivative of $f$, which we denote by $f'$, is defined in terms of the metric induced by the nonarchimedean valuation $v$, as follows. We say that $\gamma \in \operatorname{dom} f'$ and $f'(\gamma) = \alpha$ provided that

$$\lim\left(\frac{f(\gamma_n) - f(\gamma)}{\gamma_n - \gamma}\right) = \alpha$$

for any sequence $(\gamma_n)$ such that $\lim(\gamma_n) = \gamma$, $\gamma_n \in \operatorname{dom} f$, and $\gamma_n \neq \gamma$ for each $n \in N$.

6.1. EXAMPLE. Let $f$ be the function such that $f(\gamma) = \gamma^2$ for each $\gamma \in {}^\rho R$. Let $(\gamma_n)$ be any sequence such that $\lim(\gamma_n) = \gamma$, where $\gamma_n \neq \gamma$ for each $n \in N$. Then

$$\lim\left(\frac{f(\gamma_n) - f(\gamma)}{\gamma_n - \gamma}\right) = \lim\left(\frac{\gamma_n^2 - \gamma^2}{\gamma_n - \gamma}\right) = \lim(\gamma_n + \gamma) = 2\gamma$$

since $\lim(|\gamma_n - \gamma|_v) = 0$ by assumption. Thus $\gamma \in \operatorname{dom} f'$ and $f'(\gamma) = 2\gamma$ for each $\gamma \in {}^\rho R$.

6.2. THEOREM. *Let $f$ be a standard function such that $f'$ is continuous on an open interval $(a, b)$. Then ${}^\rho f$ is differentiable on ${}^\rho(a, b)$, and $({}^\rho f)' = {}^\rho(f')$.*

*Proof*. As we have seen earlier, $\operatorname{dom} {}^\rho f$ includes ${}^\rho(a,b)$. Let $\gamma \in {}^\rho(a,b)$, and let $(\gamma_n)$ be a sequence such that $\lim(\gamma_n) = \gamma$, $\gamma_n \in \operatorname{dom} f$ and $\gamma_n \neq \gamma$ for each $n \in N$. Let $x_n \in \gamma_n$ for each $n \in N$, and let $x \in \gamma$. By the Mean Value Theorem for ${}^*\mathcal{R}$, corresponding to each $n \in N$ there is a number $t_n$ between $x_n$ and $x$ such that

$$f(x_n) - f(x) = (x_n - x) f'(t_n).$$

Let $t_n \in \alpha_n$ for each $n \in N$, where $\alpha_n \in {}^\rho R$. Then for each $n \in N$,

$$\frac{{}^\rho f(\gamma_n) - {}^\rho f(\gamma)}{\gamma_n - \gamma} = {}^\rho(f')(\alpha_n).$$

Thus

$$\lim\left(\frac{{}^\rho f(\gamma_n) - {}^\rho f(\gamma)}{\gamma_n - \gamma}\right) = \lim({}^\rho(f')(\alpha_n)) = [{}^\rho(f')](\gamma)$$

since ${}^\rho(f')$ is seq-continuous at $\gamma$, by Theorem 5.6, and $\lim(\alpha_n) = \gamma$ by Lemma 3.4.9. We conclude that $\gamma \in \operatorname{dom}({}^\rho f)'$ and $({}^\rho f)'(\gamma) = [{}^\rho(f')](\gamma)$ for each $\gamma \in {}^\rho(a,b)$. Thus $({}^\rho f)' = {}^\rho(f')$ on ${}^\rho(a,b)$. This completes our proof of the theorem. □

CHAPTER 5

# EULER–MACLAURIN EXPANSIONS

## 1. Introduction

In this chapter we shall approach asymptotic expansions in an informal manner in order to motivate the intuitive idea; in Chapter 6 we shall present the formal concept of asymptotic expansions.

It is certainly true that a convergent series can be used to approximate its sum to any required accuracy; in certain cases, however, this involves summing so many terms of the series that this technique becomes impractical. We seek a practical formula (from a computing viewpoint) that yields an approximation to a given quantity (e.g., a definite integral) with only a few computations; even though the remainder term of the resulting formula cannot be made less than any given positive number by the usual device of increasing the number of terms summed.

Surprisingly, a divergent series can yield a practical formula for approximating a given quantity. Here is an example.

1.1. EXAMPLE. The *incomplete factorial* function ei (see Jeffreys and Jeffreys [1956], p.470) is defined for $t > 0$ as follows:

$$\mathrm{ei}(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx.$$

This function is closely related to the *exponential integral* function $E_i$; indeed, for each $t > 0$,

$$E_i(t) = e^t \int_t^\infty \frac{e^{-x}}{x}\,dx$$

$$= e^t\,\mathrm{ei}(t).$$

(see Bellman [1964]). We seek a method of evaluating the incomplete factorial function ei. The idea is to construct a divergent series such that the sum of the first few terms of the series approximates $\mathrm{ei}(t)$. Our method is simple – we shall integrate the given integral by parts, and repeat this procedure. Now, for each $n \in N$,

$$\int_t^\infty \frac{e^{-x}}{x^n} = -\frac{e^{-x}}{x^n}\bigg|_t^\infty - n\int_t^\infty \frac{e^{-x}}{x^{n+1}} = \frac{e^{-t}}{t^n} - n\int_t^\infty \frac{e^{-x}}{x^{n+1}} \,.$$

(Notice that we suppress $dx$ in an integral if the independent variable of the integrand, here $x$, can be determined without this aid.) So

$$\begin{aligned}
\mathrm{ei}(t) &= \frac{e^{-t}}{t} - \int_t^\infty \frac{e^{-x}}{x^2} \\
&= e^{-t}\left[\frac{1}{t} - \frac{1}{t^2}\right] + 2\int_t^\infty \frac{e^{-x}}{x^3} \\
&= e^{-t}\left[\frac{1}{t} - \frac{1}{t^2} + \frac{2!}{t^3}\right] - 3!\int_t^\infty \frac{e^{-x}}{x^4} \\
&= e^{-t}\left[\frac{1}{t} - \frac{1}{t^2} + \ldots + (-1)^{n-1}\frac{(n-1)!}{t^n}\right] + (-1)^n n!\int_t^\infty \frac{e^{-x}}{x^{n+1}} .
\end{aligned}$$

Denoting the last term on the right by "$R$", we point out that

$$(1.2) \qquad |R| < \frac{n!}{t^{n+1}}\int_t^\infty e^{-x} = \frac{n!\,e^{-t}}{t^{n+1}} = \frac{1}{t}\frac{2}{t}\cdots\frac{n}{t}\frac{e^{-t}}{t} .$$

For a fixed $t$, the product on the right is minimized by taking $n$ to be the integral part of $t$. For example, let us approximate $\mathrm{ei}(5)$; accordingly, we take $n = 5$; i.e., we compute the sum of the first five terms of the divergent series

$$e^{-5}\sum_{n\in N}(-1)^n\frac{n!}{5^{n+1}} .$$

So

$$\begin{aligned}
\mathrm{ei}(5) &\doteq e^{-5}\left(\tfrac{1}{5} - \tfrac{1}{25} + \tfrac{2}{125} - \tfrac{6}{625} + \tfrac{24}{3125}\right) \\
&\doteq .006\,738 \times .174\,1 \\
&\doteq .001\,17.
\end{aligned}$$

Here

$$|R| < 5!\,\frac{e^{-5}}{5^6} \doteq \tfrac{24}{3125} \times .006\,7 \doteq .000\,052.$$

Since the remainder term, in this case, is negative, we conclude that ei(5) = .001 1... to four decimal places. The correct value, to five decimal places, is .001 15. Notice that our approximation to ei(5) is *not* improved by taking more terms of the divergent series $e^{-5}\sum_{n\in N}(-1)^n n!/5^{n+1}$; indeed, the sum of the first seven terms of this series is

$$e^{-5}\left(\tfrac{1}{5} - \tfrac{1}{25} + \tfrac{2}{125} - \tfrac{6}{625} + \tfrac{24}{3125} - \tfrac{24}{3125} + \tfrac{144}{15625}\right) \doteq 6.737\,9 \times .175\,62 \times 10^{-3}$$

$$\doteq .001\,18.$$

This is typical of asymptotic expansions; there is a natural number, say $q$, such that summing terms of the expansion up to the $q^{\text{th}}$ term improves the approximation, whereas including terms beyond the $q^{\text{th}}$ term worsens the approximation.

Another typical feature of asymptotic expansions is that the accuracy of an approximation depends on the size of the argument $t$. For example, we can compute ei(10) with an error less than $1.7 \times 10^{-9}$ by summing the first ten terms of the expansion given above; we find that ei(10) ≐ .00000416. Incidentally, now that we know the value of ei(10) to eight decimal places, we can compute ei($t$), where $t$ is close to 10, by other methods; we rely on the fact that

$$\mathrm{ei}(t) = \mathrm{ei}(10) + \int_t^{10} \frac{e^{-x}}{x}.$$

For example, using the quadrature formula: for some $c \in (a-h, a+h)$,

$$\int_{a-h}^{a+h} f = \tfrac{1}{5}h[f(a-h) + f(a+h)] + \tfrac{8}{5}h\,f(a) + \tfrac{2}{15}h^3 f^{(2)}(a) - \tfrac{1}{6300}h^7 f^{(6)}(c)$$

(see Lightstone [1966]), we find that

$$\int_{9.8}^{10} \frac{e^{-x}}{x} \doteq .000\,001\,02.$$

So

$$\mathrm{ei}(9.8) \doteq .000\,004\,16 + .000\,001\,02 = .000\,005\,18.$$

In this way we can build up a table of values of the function ei.

Turning to the *exponential integral* function $E_i$, we have that for each $n \in N$ and for each $t > 0$,

$$E_i(t) \doteq \frac{1}{t} - \frac{1}{t^2} + \ldots + (-1)^n \frac{n!}{t^{n+1}}.$$

From (1.2), the error in this approximation is $R$, where

$$|R| < \frac{(n+1)!}{t^{n+2}}.$$

This bound on the error is minimized by taking $n$ so that $n + 1 = [t]$, the integral part of $t$.

## 2. The Euler–Maclaurin Formula

The Euler–Maclaurin Formula 2.7 is a fertile source of asymptotic expansions; we shall illustrate this fact in Section 3. The purpose of this section is to derive this formula. We shall follow the method of Jeffreys and Jeffreys [1956].

Let $f$ be a function such that the closed interval $[0, 1]$ is a subset of $\operatorname{dom} f^{(2r+1)}$, where $r \in N$. Consider the problem of computing $\int_0^1 f$; integrating by parts we obtain

$$(2.1) \qquad \int_0^1 f = (x - \tfrac{1}{2}) f \Big|_0^1 - \int_0^1 (x - \tfrac{1}{2}) f' = \tfrac{1}{2}[f(0) + f(1)] - \int_0^1 (x - \tfrac{1}{2}) f'.$$

We wish to continue integrating by parts. This operation is expedited by introducing functions $P_2, P_3, P_4, \ldots$, each with domain including $[0,1]$ and such that

(a) $P_2' = x - \frac{1}{2}$;

(b) $P_n(0) = 0$ for each $n > 1$;

(c) $P_{n+1}' = b_n + P_n$, where $b_n$ is chosen so that $P_n(1) = 0$ for each $n > 1$.

The functions $P_i$ and numbers $b_i$, $i > 1$, are generated one after the other by (a), (b) and (c). For example, $P_2 = \frac{1}{2}x^2 - \frac{1}{2}x$, $P_3 = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x$, and $b_2 = \frac{1}{12}$.

Our immediate goal, however, is to derive the quadrature formula (2.5).

To this end, let $n > 1$; then

$$\int_0^1 P_n f^{(n)} = \int_0^1 (P'_{n+1} - b_n) f^{(n)} \qquad \text{by (c)}$$

$$= -b_n f^{(n-1)}\Big|_0^1 + P_{n+1} f^{(n)}\Big|_0^1 - \int_0^1 P_{n+1} f^{(n+1)}$$

$$= -b_n f^{(n-1)}\Big|_0^1 - \int_0^1 P_{n+1} f^{(n+1)}.$$

Thus

$$(2.2)\quad \int_0^1 (x-\tfrac{1}{2}) f' = \int_0^1 P'_2 f' = P_2 f'\Big|_0^1 - \int_0^1 P_2 f'' = -\int_0^1 P_2 f''$$

$$= b_2 f'\Big|_0^1 + \int_0^1 P_3 f^{(3)} \qquad \text{by the preceding result}$$

$$= b_2 f'\Big|_0^1 - b_3 f''\Big|_0^1 + \ldots + (-1)^r b_r f^{(r-1)}\Big|_0^1 + (-1)^r \int_0^1 P_{r+1} f^{(r+}$$

We want to prove that $b_i = 0$ if $i$ is odd, $i > 2$. To this purpose, let $c_0, c_1, c_2, \ldots$ and $Q_0, Q_1, Q_2, \ldots$ be numbers and functions such that for each $a \in R$,

$$(2.3)\qquad \frac{a}{e^a - 1} + \tfrac{1}{2}a = \sum_N c_i a^i,$$

$$(2.4)\qquad \frac{a(e^{ax} - 1)}{e^a - 1} = \sum_N a^i Q_i.$$

Differentiating (2.4),

$$\sum_N a^i Q'_i = \frac{a^2 e^{ax}}{e^a - 1} = \sum_N a^{i+1} Q_i + \frac{a^2}{e^a - 1}$$

$$= \sum_N a^{i+1} Q_i + \sum_N c_i a^{i+1} - \tfrac{1}{2}a^2 \qquad \text{by (2)}$$

i.e.,

$$\sum_N a^i Q'_i = \sum_N a^{i+1}(Q_i + c_i) - \tfrac{1}{2}a^2.$$

Thus, for each $i > 1$, $Q'_{i+1} = Q_i + c_i$.

From (2.4), for each $a \in R$,

$$(a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots)(Q_0 + aQ_1 + a^2Q_2 + \ldots) = a(ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \ldots),$$

so $Q_0 = 0$, $Q_1 = x$, and $Q_2 = \frac{1}{2}x^2 - \frac{1}{2}x$.

Evaluating (2.4) at 0 yields

$$\sum_N a^i Q_i(0) = 0;$$

so $Q_i(0) = 0$ for each $i \in N$. Evaluating (2.4) at 1 yields

$$\sum_N a^i Q_i(1) = a;$$

so $Q_i(1) = 0$ if $i \neq 1$, and $Q_1(1) = 1$. We conclude that $Q_i = P_i$ and $c_i = b_i$ for each $i > 1$. This means that (2.3) characterizes the $b$'s and (2.4) characterizes the $P$'s.

We now show that the LHS of (2.3) is an even function of $a$:

$$-\frac{a}{e^{-a} - 1} - \tfrac{1}{2}a = \frac{a}{1 - e^{-a}} - \tfrac{1}{2}a = \frac{a e^a}{e^a - 1} - \tfrac{1}{2}a = a + \frac{a}{e^a - 1} - \tfrac{1}{2}a$$

$$= \frac{a}{e^a - 1} + \tfrac{1}{2}a,$$

which is the LHS of (2.3). Therefore, the RHS of (2.3) is an even function of $a$, i.e.,

$$\sum_N c_i(-a)^i = \sum_N c_i a^i,$$

so

$$2(c_1 a + c_3 a^3 + c_5 a^5 + \ldots) = 0.$$

We conclude that $c_i = 0$ if $i$ is odd; so $b_i = 0$ if $i$ is odd. This is an important observation for our purposes; not only does it allow us to simplify (2.2), but it helps us deduce an important property of the remainder term $R$ in the Euler–Maclaurin Formula, as we shall later see. From (2.2), for each $m \geq 1$,

$$\int_0^1 (x - \tfrac{1}{2})f' = \sum_1^m b_{2i} f^{(2i-1)}\Big|_0^1 + \int_0^1 P_{2m+1} f^{(2m+1)}.$$

So, from (2.1), we obtain the following quadrature formula. For each $m \geqslant 1$,

$$\int_0^1 f = \tfrac{1}{2}[f(0)+f(1)] - \sum_1^m b_{2i} f^{(2i-1)}\Big|_0^1 - \int_0^1 P_{2m+1} f^{(2m+1)}. \tag{2.5}$$

It is convenient to switch over to Bernoulli numbers $B_i$ and Bernoulli polynomials $\phi_i$ which can be defined as follows: for each $i \in N$,

$$B_i = i!\, c_i, \qquad \phi_i = i!\, Q_i.$$

For later reference, we list some of these numbers and polynomials:

$$\begin{aligned} &B_0 = 1, && B_2 = \tfrac{1}{6}, && B_4 = -\tfrac{1}{30},\\ &B_6 = \tfrac{1}{42}, && B_8 = -\tfrac{1}{30}, && B_{10} = \tfrac{5}{66},\\ &B_{12} = -\tfrac{691}{2730}, && B_{14} = \tfrac{7}{6}, && B_{16} = -\tfrac{3617}{510}; \end{aligned}$$

and

$$\begin{aligned} &\phi_0 = 0, && \phi_1 = x,\\ &\phi_2 = x^2 - x, && \phi_3 = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,\\ &\phi_4 = x^4 - 2x^3 + x^2, && \phi_5 = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x,\\ &\phi_6 = x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2, && \phi_7 = x^7 - \tfrac{7}{2}x^6 + \tfrac{7}{2}x^5 - \tfrac{7}{6}x^3 + \tfrac{1}{6}x,\\ &\phi_8 = x^8 - 4x^7 + \tfrac{14}{3}x^6 - \tfrac{7}{3}x^4 + \tfrac{2}{3}x^2, && \phi_9 = x^9 - \tfrac{9}{2}x^8 + 6x^7 - \tfrac{21}{5}x^5 + 2x^3 - \tfrac{3}{10}x. \end{aligned}$$

In terms of Bernoulli numbers and Bernoulli polynomials the *quadrature formula* (2.5) can be stated as follows. For each $r \geqslant 1$,

$$\int_0^1 f = \tfrac{1}{2}[f(0)+f(1)] - \sum_1^r \frac{B_{2i}}{(2i)!} f^{(2i-1)}\Big|_0^1 - \frac{1}{(2r+1)!}\int_0^1 \phi_{2r+1} f^{(2r+1)}. \tag{2.6}$$

For each $i \in N$,

$$\int_i^{i+1} f = \int_0^1 f(x+i);$$

so, for each $n \in N$,

$$\int_0^n f = \sum_{i=0}^{n} \int_0^1 f(x+i).$$

This observation, together with the quadrature formula (2.6), yields the Euler–Maclaurin Formula which we now state.

2.7. EULER–MACLAURIN FORMULA. *For each* $n \in N$,

$$\int_0^n f = \tfrac{1}{2}[f(0) + f(n)] + \sum_1^{n-1} f(i) - \sum_1^r \frac{B_{2i}}{(2i)!} f^{(2i-1)}\Big|_0^n + R,$$

*where*

$$R = -\frac{1}{(2r+1)!} \sum_0^{n-1} \int_i^{i+1} \phi_{2r+1}(x-1) f^{(2r+1)}.$$

We want to show that if, for each $r$, $f^{(2r+1)}$ is monotonic on $[0, n]$, and the values of $f^{(2r+2)}$ and $f^{(2r+4)}$ have the same sign throughout $[0, n]$, then the remainder term $R$ of the Euler–Maclaurin Formula changes sign when $r$ is increased by one. In this case, then, the error in approximating $\int_0^n f$ by a finite number of terms of the *Euler–Maclaurin expansion* (see (2.11)) is bounded by the absolute value of the next term of the expansion; i.e.,

$$|R| < \left|\frac{B_{2r+2}}{(2r+2)!} [f^{(2r+1)}(n) - f^{(2r+1)}(0)]\right|. \tag{2.8}$$

Thus

$$\int_0^n f \doteq \tfrac{1}{2} f(0) + \tfrac{1}{2} f(n) + \sum_i^{n-1} f(i) - \sum_1^r \frac{B_{2i}}{(2i)!} f^{(2i-1)}\Big|_0^n,$$

and the error in this approximation is bounded by the RHS of (2.8).

With this goal in mind, we point out that on $[0, 1]$ the zeros of $P_{2r}$ are 0 and 1, and the zeros of $P_{2r+1}$ are $0, \frac{1}{2}$ and 1 (see Jeffreys and Jeffreys [1956]). Therefore the values of $P_{2r}$ have the same sign on the open interval $(0, 1)$, and the values of $P_{2r+1}$ have the same sign on the open interval $(0, \frac{1}{2})$.

From (c),

$$\int_0^1 P'_{n+1} = b_n + \int_0^1 P_n;$$

so

$$b_n = -\int_0^1 P_n$$

for each $n > 1$. Thus

$$\operatorname{sgn} P_{2r} \neq \operatorname{sgn} b_{2r},$$

where

$$\operatorname{sgn} P_i = \begin{cases} + & \text{if the values of } P_i \text{ on } (0, \tfrac{1}{2}) \text{ are all positive,} \\ - & \text{if the values of } P_i \text{ on } (0, \tfrac{1}{2}) \text{ are all negative,} \end{cases}$$

and

$$\int_0^1 P_{2r+1} = 0$$

since $b_{2r+1} = 0$. So the values of $P_{2r+1}$ on $(0, \frac{1}{2})$ are positive iff the values of $P_{2r+1}$ on $(\frac{1}{2}, 1)$ are negative.

Let $0 < h < \frac{1}{2}$; by the Mean Value Theorem there is a number $t$, $0 < t < h$, such that

$$P_n(h) - P_n(0) = h P_n'(t);$$

i.e., $P_n(h) = h P_n'(t)$. So

$$\operatorname{sgn} P_n = \operatorname{sgn} P_n'$$

for each $n > 1$. Moreover, from (c) and the fact that $b_i = 0$ if $i$ is odd, we obtain that

$$P_{2r}' = P_{2r-1}$$

for each $r > 1$. By the Mean Value Theorem,

$$P_{2r}(h) = h P_{2r}'(t) = h P_{2r-1}(t)$$

for some $t \in (0, h)$. Thus

$$\operatorname{sgn} P_{2r} = \operatorname{sgn} P_{2r-1}.$$

Since $P_{2r}(0) = 0$, we can choose $h \in (0, \frac{1}{2})$ so that

$$|P_{2r}(h)| < |b_{2r}|;$$

thus $P_{2r+1}'(h) = b_{2r} + P_{2r}(h)$ has the same sign as $b_{2r}$, so

$$\operatorname{sgn} P_{2r+1}' = \operatorname{sgn} b_{2r}, \qquad \operatorname{sgn} P_{2r+1} = \operatorname{sgn} b_{2r}.$$

Summarizing,

$$\operatorname{sgn} P_{2r+1} = \operatorname{sgn} b_{2r} \neq \operatorname{sgn} P_{2r} = \operatorname{sgn} P_{2r-1}.$$

Thus $\operatorname{sgn} P_{2r+1} \neq \operatorname{sgn} P_{2r-1}$ (as well, $\operatorname{sgn} b_{2r} \neq \operatorname{sgn} b_{2r+2}$). This proves that

$$\operatorname{sgn} \phi_{2r+1} \neq \operatorname{sgn} \phi_{2r-1}$$

for each $r > 1$.

To establish our assertion (2.8) we must consider the remainder term $R$ of the Euler–Maclaurin Formula. First, notice that for each $i$,

$$\int_i^{i+1} P_{2r+1}(x-i)\,[f^{(2r+1)} - f^{(2r+1)}(i+\tfrac{1}{2})]$$

$$= \int_i^{i+1} P_{2r+1}(x-i)\,f^{(2r+1)} - f^{(2r+1)}(i+\tfrac{1}{2}) \int_i^{i+1} P_{2r+1}(x-i)$$

$$= \int_i^{i+1} P_{2r+1}(x-i)\,f^{(2r+1)} - f^{(2r+1)}(i+\tfrac{1}{2}) \int_0^1 P_{2r+1}$$

$$= \int_i^{i+1} P_{2r+1}(x-i)\,f^{(2r+1)}$$

since

$$\int_0^1 P_{2r+1} = \int_0^1 (P'_{2r+2} - b_{2r+1}) = \int_0^1 P'_{2r+2} = P_{2r+2}\Big|_0^1 = 0.$$

Therefore

$$\int_i^{i+1} \phi_{2r+1}(x-i)\,f^{(2r+1)} = \int_i^{i+1} \phi_{2r+1}(x-i)\,[f^{(2r+1)} - f^{(2r+1)}(i+\tfrac{1}{2})].$$

Bear in mind that for each $r \geqslant 1$, the values of $\phi_{2r+1}$ on the interval $(0, \frac{1}{2})$ have the opposite sign to the values of $\phi_{2r+1}$ on $(\frac{1}{2}, 1)$. Let $f^{(2r+1)}$ be monotonic on $[0, n]$; then for each $i$ the values of $f^{(2r+1)} - f^{(2r+1)}(i+\frac{1}{2})$ on $(i, i+\frac{1}{2})$ have the opposite sign to the values of $f^{(2r+1)} - f^{(2r+1)}(i+\frac{1}{2})$ on the interval $(i+\frac{1}{2}, i+1)$, or else are zero (but zero values can safely be ignored). Moreover, for each $i$, and fixed $r$, the values of $f^{(2r+1)} - f^{(2r+1)}(i+\frac{1}{2})$ on $(i, i+\frac{1}{2})$ have the same sign. This means that for fixed $r$, the sign of

$$\int_i^{i+1} \phi_{2r+1}(x-i)\,[f^{(2r+1)} - f^{(2r+1)}(i+\tfrac{1}{2})]$$

is the same for each $i$.

Finally, suppose that both $f^{(2r+1)}$ and $f^{(2r+3)}$ are increasing on $[0, n]$, or both functions are decreasing on $[0, n]$. We claim that in this case the remainder term of the Euler–Maclaurin Formula changes sign when $r$ is increased by one. Let $R_1$ be the first remainder term and $R_2$ the second remainder term. The sign of $R_1$ is the sign of

$$-\int_i^{i+1} \phi_{2r+1}(x-i)\,[f^{(2r+1)} - f^{(2r+1)}(i+\tfrac{1}{2})];$$

the sign of $R_2$ is the sign of

$$-\int_i^{i+1} \phi_{2r+3}(x-i)\,[f^{(2r+3)} - f^{(2r+3)}(i+\tfrac{1}{2})].$$

But $\operatorname{sgn}\phi_{2r+1} \neq \operatorname{sgn}\phi_{2r+3}$; also, the values of $f^{(2r+1)} - f^{(2r+1)}(i+\frac{1}{2})$ on $(i, i+\frac{1}{2})$ have the same sign as the values of $f^{(2r+3)} - f^{(2r+3)}(i+\frac{1}{2})$ on $(i, i+\frac{1}{2})$. We conclude that $R_1$ and $R_2$ have opposite signs. Thus

$$\int_0^n f = a + R_1, \qquad \int_0^n f = a + b + R_2,$$

where $R_1$ and $R_2$ have opposite signs; so $b = R_1 - R_2$ and the three numbers $b, R_1$ and $-R_2$ have the same sign. It follows that $|R_1| < |b|$. Of course, $f^{(2r+1)}$ and $f^{(2r+3)}$ are both increasing, or both decreasing, on $[0, n]$ if the values of $f^{(2r+2)}$ and $f^{(2r+4)}$ have the same sign on $[0, n]$. This establishes (2.8).

In the examples of Section 3 we shall be concerned with approximating definite integrals of the form $\int_a^n f$, where $a \in I$. To apply the Euler–Maclaurin Formula here, observe again that

$$\int_a^n f = \int_0^{n-a} f(x+a),$$

which is a direct consequence of the characterization of a definite integral in terms of a sum. Applying the Euler–Maclaurin Formula to $\int_0^{n-a} f(x+a)$ yields for each $r$,

$$\int_a^n f = \tfrac{1}{2}f(a) + \tfrac{1}{2}f(n) + \sum_{a+1}^{n-1} f(i) - \sum_1^r \frac{B_{2i}}{(2i)!} f^{(2i-1)}\Big|_a^n + R, \tag{2.9}$$

where

$$R = -\frac{1}{(2r+1)!} \sum_0^{n-1} \int_{a+i}^{a+i+1} \phi_{2r+1}(x-a-i)\, f^{(2r+1)}.$$

We mention that (2.9) is also known as an Euler–Maclaurin Formula.

Moreover, if for each $r$, $f^{(2r+1)}$ is monotonic on $[a, n]$, and the values of $f^{(2r+2)}$ and $f^{(2r+4)}$ have the same sign throughout $[a, n]$, then

$$|R| < \left| \frac{B_{2r+2}}{(2r+2)!} [f^{(2r+1)}(n) - f^{(2r+1)}(a)] \right| , \tag{2.10}$$

i.e., $|R|$ is bounded by the absolute value of the next term of the expansion.

Let

$$c = \tfrac{1}{2} f(a) + \tfrac{1}{2} f(n) + \sum_{a+1}^{n-1} f(i),$$

and consider the expression

$$c - \sum \frac{B_{2i}}{(2i)!} f^{(2i-1)} \Big|_a^n , \tag{2.11}$$

which is known as the *Euler–Maclaurin expansion* of $\int_a^n f$. We express this by writing

$$\int_a^n f \mathrel{\dot\sim} c - \sum \frac{B_{2i}}{(2i)!} f^{(2i-1)} \Big|_a^n$$

or simply

$$\int_a^n f \mathrel{\dot\sim} c - \frac{B_2}{2!} f' \Big|_a^n - \frac{B_4}{4!} f^{(3)} \Big|_a^n - \frac{B_6}{6!} f^{(5)} \Big|_a^n - \dots .$$

For each $r \in N$, the number

$$c - \sum_1^r \frac{B_{2i}}{(2i)!} f^{(2i-1)} \Big|_a^n$$

is called an *Euler–Maclaurin approximation* to $\int_a^n f$; as claimed earlier, this yields

$$\int_a^n f \doteq c - \sum_1^r \frac{B_{2i}}{(2i)!} f^{(2i-1)} \Big|_a^n .$$

Of course, the accuracy of this approximation depends upon the remainder term $R$ of the Euler–Maclaurin Formula (2.9).

### 3. Some examples

As we have seen, we can approximate $\int_a^n f$ by the Euler–Maclaurin Formula (2.9). However, to ensure that (2.10) provides a bound on the remainder term $R$, we require that:

(1) for each odd $i$, $f^{(i)}$ is monotonic on $[a, n]$;

(2) the values of $f^{(2r+2)}$ and $f^{(2r+4)}$ have the same sign throughout $[a, n]$.

In this case, then, we are in the fortunate position that the absolute value of the error in the Euler–Maclaurin approximation

$$(3.1)\qquad \int_a^n f \doteq \tfrac{1}{2}f(a) + \tfrac{1}{2}f(n) + \sum_{a+1}^{n-1} f(i) - \sum_1^r \frac{B_{2i}}{(2i)!} f^{(2i-1)}\Big|_a^n$$

is less than the absolute value of the next term of the expansion.

We now illustrate this result.

3.2. EXAMPLE. The function $1/x$ satisfies conditions (1) and (2) since

$$(1/x)^{(i)} = (-1)^i i!\, x^{-i-1}$$

for each $i$. Thus, for $a > 0$ and $i$ odd, $(1/x)^{(i)}$ is monotonic on $[a, n]$, and the functions $(1/x)^{(2r+2)}$ and $(1/x)^{(2r+4)}$ are positive on $[a, n]$. So, by (3.1),

$$\ln 2 = \int_5^{10} \frac{1}{x} \doteq \tfrac{1}{10} + \tfrac{1}{20} + (\tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9}) + \tfrac{1}{2}B_2 x^{-2}\Big|_5^{10} + \tfrac{1}{4}B_4 x^{-4}\Big|_5^{10} + \tfrac{1}{6}B_6 x^{-6}\Big|_5^{10}$$

$$\doteq .695\,634\,92 - \tfrac{1}{12}(5^{-2} - 10^{-2}) + \tfrac{1}{120}(5^{-4} - 10^{-4}) - \tfrac{1}{252}(5^{-6} - 10^{-6})$$

$$\doteq .695\,634\,92 - .002\,487\,75 = .693\,147\,17.$$

The next term in our expansion is

$$\tfrac{1}{8}B_8 x^{-8}\Big|_5^{10} \doteq 10^{-8};$$

so we anticipate that our approximation to ln 2 is slightly too small, with an error of about one in the eighth decimal place. In fact, ln 2 = .693 147 180 .... We can compute ln 5 to seven decimal places with only a few calculations, as follows:

$$\ln 5 = \int_4^{20} \frac{1}{x} = \int_4^{16} \frac{1}{x} + \int_{16}^{20} \frac{1}{x} = 2\ln 2 + \int_{16}^{20} \frac{1}{x}$$

$$\doteq 1.386\,294\,36 + \tfrac{1}{32} + \tfrac{1}{40} + \left(\tfrac{1}{17} + \tfrac{1}{18} + \tfrac{1}{19}\right) + \tfrac{1}{2}B_2 x^{-2}\Big|_{16}^{20} + \tfrac{1}{4}B_4 x^{-4}\Big|_{16}^{20}$$

$$\doteq 1.609\,437\,91\,.$$

Notice that

$$\tfrac{1}{6}B_6 x^{-6}\Big|_{16}^{20} = -\tfrac{1}{252}(16^{-6} - 20^{-6});$$

so the error in absolute value is less than $3 \times 10^{-10}$. We mention that $\ln 5 = 1.609\,437\,912\,4\ldots$.

In our next example we use the Euler–Maclaurin approximation (3.1) to approximate Euler's constant $\gamma$.

3.3. EXAMPLE. For each natural number $n$,

$$\ln n - \ln 5 = \int_5^{n} \frac{1}{x} \doteq \frac{1}{10} + \frac{1}{2n} + \sum_{6}^{n-1} \frac{1}{i} + \frac{B_2}{2} x^{-2}\Big|_5^n + \frac{B_4}{4} x^{-4}\Big|_5^n + \frac{B_6}{6} x^{-6}\Big|_5^n,$$

thus

$$\lim\left(\sum_{6}^{n-1} \frac{1}{i} + \ln 5 - \ln n\right) \doteq -\frac{1}{10} + \frac{5^{-2}}{12} - \frac{5^{-4}}{120} + \frac{5^{-6}}{252} \doteq -.096\,679\,75.$$

Now

$$\gamma = \lim(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{n} - \ln n)$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \ln 5 + \lim(\tfrac{1}{6} + \ldots + \tfrac{1}{n} - \ln n + \ln 5)$$

$$\doteq 2.283\,333\,33 - 1.609\,437\,91 - .096\,679\,75$$

$$= .577\,215\,67.$$

In fact, $\gamma = .577\,215\,664\,9\ldots$.

To see the importance of Euler's constant, notice how we use $\gamma$ to develop an improved asymptotic expansion for $\ln t$.

3.4. EXAMPLE. By the Euler–Maclaurin formula,

$$\ln n - \ln t = \int_t^n \frac{1}{x} \dot{\sim} \frac{1}{2t} + \frac{1}{2n} + \sum_{t+1}^{n-1} \frac{1}{i} + \frac{B_2}{2} x^{-2}\Big|_t^n + \frac{B_4}{4} x^{-4}\Big|_t^n + \ldots$$

so

$$-\ln t \dot{\sim} \frac{1}{2t} + \lim\Big(\sum_1^n \frac{1}{i} - \ln n\Big) - \sum_1^t \frac{1}{i} - \frac{B_2}{2t^2} - \frac{B_4}{4t^4} - \ldots$$

thus

$$\ln t \dot{\sim} \sum_1^t \frac{1}{i} - \frac{1}{2t} - \gamma + \frac{B_2}{2t^2} + \frac{B_4}{4t^4} + \ldots \tag{3.5}$$

For example,

$$\ln 10 \dot{\sim} \sum_1^{10} \frac{1}{i} - \frac{1}{20} - \gamma + \frac{B_2}{2} 10^{-2} + \frac{B_4}{4} 10^{-4} + \ldots,$$

so

$$\ln 10 \doteq 2.928\,968\,253\,97 - .627\,215\,664\,9 + \tfrac{1}{12} \times 10^{-2} - \tfrac{1}{120} \times 10^{-4} + \tfrac{1}{252} \times 10^{-6}$$

$$\doteq 2.301\,752\,589\,07 + .000\,832\,503\,97$$

$$\doteq 2.302\,585\,093\,0.$$

In fact, $\ln 10 = 2.302\,585\,092\,994\ldots$ .

From (3.5) we easily obtain an asymptotic expansion of $\Sigma_{t+1}^{2t} 1/i$, as follows.

3.6. EXAMPLE. Taking the difference of the asymptotic expansions of $\ln 2t$ and $\ln t$ yielded by (3.5), we obtain

$$\ln 2t - \ln t \dot{\sim} -\frac{1}{4t} + \frac{1}{2t} + \sum_{t+1}^{2t} \frac{1}{i} + \frac{B_2}{2}\left(\frac{1}{4t^2} - \frac{1}{t^2}\right) + \frac{B_4}{4}\left(\frac{1}{16t^4} - \frac{1}{t^4}\right) + \ldots$$

$$\ln 2 \dot{\sim} \frac{1}{4t} + \sum_{t+1}^{2t} \frac{1}{i} - \frac{B_2}{2 \cdot 2^2}(2^2 - 1)t^{-2} - \frac{B_4}{4 \cdot 2^4}(2^4 - 1)t^{-4} - \ldots$$

so

$$\sum_{t+1}^{2t} \frac{1}{i} \dot{\sim} \ln 2 - \frac{1}{4t} + B_2 \frac{2^2 - 1}{2 \cdot 2^2} t^{-2} + B_4 \frac{2^4 - 1}{4 \cdot 2^4} t^{-4} + \ldots, \tag{3.7}$$

To illustrate,

$$\sum_{11}^{20}\frac{1}{i} \doteq .693\,147 - .025 + .000\,625 = .668\,772.$$

In fact, $\Sigma_{11}^{20}\ 1/i = .668\,771\,4\ldots$.

In a moment we shall derive an asymptotic expansion for $\frac{1}{6}\pi^2$. To illustrate the idea we first compute $(\frac{1}{101})^2 + \ldots + (\frac{1}{200})^2$ to ten decimal places.

3.8. EXAMPLE. By the Euler–Maclaurin formula,

$$\int_{100}^{200}\frac{1}{x^2} \doteq \frac{1}{2\cdot 10^4} + \frac{1}{8\cdot 10^4} + \sum_{101}^{199}\frac{1}{i^2} + B_2 x^{-3}\Big|_{100}^{200}$$

$$\doteq \sum_{101}^{199}\frac{1}{i^2} + .000\,062\,354\,2.$$

So

$$\sum_{101}^{200}\frac{1}{i^2} \doteq .000\,025 - .000\,062\,354\,2 + \int_{100}^{200}\frac{1}{x^2}$$

$$\doteq -.000\,037\,354\,2 + .005$$

$$\doteq .004\,962\,645\,8.$$

Here is our computation for $\frac{1}{6}\pi^2$ and $\pi$.

3.9. EXAMPLE. For each infinite natural number $\kappa$,

$$\int_{10}^{\kappa}\frac{1}{x^2} \doteq .005 + \sum_{11}^{\kappa-1}\frac{1}{i^2}$$

$$+ B_2 x^{-3}\Big|_{10}^{\kappa} + B_4 x^{-5}\Big|_{10}^{\kappa} + B_6 x^{-7}\Big|_{10}^{\kappa} + B_8 x^{-9}\Big|_{10}^{\kappa} + B_{10} x^{-11}\Big|_{10}^{\kappa}$$

$$\simeq .005 + \sum_{11}^{\kappa-1}\frac{1}{i^2} + B_2\cdot 10^{-3} + B_4\cdot 10^{-5}$$

$$- B_6\cdot 10^{-7} - B_8\cdot 10^{-9} - B_{10}\cdot 10^{-11}.$$

So

$$\sum_N \frac{1}{i^2} \simeq \sum_1^{10} \frac{1}{i^2} + \sum_{11}^{k-1} \frac{1}{i^2}$$

$$\dot{\sim} \sum_1^{10} \frac{1}{i^2} + .095 + B_2 \cdot 10^{-3} + B_4 \cdot 10^{-5} + \ldots$$

(our asymptotic expansion)

$$\doteq 1.549\,767\,731\,166\,4 + .095\,166\,335\,681\,8$$

$$= 1.644\,934\,066\,848\,2$$

$$\doteq \tfrac{1}{6}\pi^2.$$

Thus

$$\pi^2 \doteq 9.869\,604\,401\,089\,2, \qquad \pi \doteq 3.141\,592\,653\,589;$$

in fact, $\pi = 3.141\,592\,653\,589\ldots$.

Our next example, although it is not an asymptotic expansion, illustrates the power of the Euler–Maclaurin Formula 2.7.

3.10. EXAMPLE. Let $p$ be any natural number. Then for each natural number $n$,

$$\int_0^n x^p = \frac{n^p}{2} + \sum_{i=1}^{n-1} i^p$$

$$- \frac{B_2}{2}\binom{p}{1} n^{p-1} - \frac{B_4}{4}\binom{p}{3} n^{p-3} - \frac{B_6}{6}\binom{p}{5} n^{p-5} - \ldots - \frac{B_q}{q}\binom{p}{q-1} n^{p-q+1},$$

where $q$ is $p$ or $p-1$. Thus

$$\sum_{i=1}^n i^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{B_2}{2}\binom{p}{1} n^{p-1} + \frac{B_4}{4}\binom{p}{3} n^{p-3} + \ldots + \frac{B_q}{q}\binom{p}{q-1} n^{p-q+1},$$

where $q$ is $p$ or $p-1$. For example,

$$\sum_1^n i^5 = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2,$$

$$\sum_1^n i^6 = \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n.$$

A characteristic feature of an asymptotic expansion, in the case of a quantity that involves a parameter, e.g. a function, is that the accuracy of the calculation depends upon the value of the parameter. In general, the accuracy improves as the parameter takes on its limiting value. For example, the greater $t$ is, the more accurately we can calculate $\mathrm{ei}(t)$ by the asymptotic expansion of Example 1.1. Here is another example that illustrates this point.

3.11. EXAMPLE. Let $s \in N$, and let $p > 1$, $p \in R$. Considering $\int_s^\infty x^{-p}$ and applying the method of Example 3.10, we obtain the asymptotic expansion

$$\sum_{i=s+1}^{\infty} i^{-p} \sim \frac{s^{1-p}}{p-1} - \frac{s^{-p}}{2} + \frac{B_2}{2}\binom{p}{1} s^{-p-1} + \frac{B_4}{4}\binom{p+2}{3} s^{-p-3} + \frac{B_6}{6}\binom{p+4}{5} s^{-p-5} + \ldots \tag{3.12}$$

The RHS of (3.12) generates approximations to its LHS by taking partial sums. For example,

$$\sum_{i=s+1}^{\infty} i^{-p} \doteq \frac{s^{1-p}}{p-1} - \frac{s^{-p}}{2} + \frac{pB_2}{2} s^{-p-1}. \tag{3.13}$$

The error in this approximation, in absolute value, is less than

$$\frac{|B_4|}{4}\binom{p+2}{3} s^{-p-3}.$$

By increasing $s$, we decrease the bound on our error. In this sense, we can approximate

$$\sum_N i^{-p} = \sum_{i=1}^{s} i^{-p} + \sum_{i=s+1}^{\infty} i^{-p}$$

as accurately as we wish by choosing $s$ sufficiently large.

Here is a famous asymptotic expansion – Stirling's formula for $\ln \Gamma(t+1)$.

3.14. EXAMPLE. For $t > -1$ and $t \in R$,

$$\Gamma(t+1) = \int_0^\infty \mathrm{e}^{-x} x^t = \lim \left( \frac{n!\, n^t}{(t+1)\ldots(t+n)} \right),$$

so

$$\ln \Gamma(t+1) = \lim(\ln n! + t \ln n - \sum_1^n \ln(t+i))$$

and

$$\text{(3.15)} \qquad \mathrm{D}\ln\Gamma(t+1) = \lim(\ln n - \sum_{1}^{n} \frac{1}{t+i}).$$

We get a grip on the RHS of (3.15) by applying the Euler–Maclaurin formula (2.9) to $\int_t^{t+n} 1/x$. This yields:

$$\text{(3.16)} \qquad \ln(t+n) - \ln t \mathrel{\dot\sim} \frac{1}{2t} + \frac{1}{2(t+n)} + \sum_{t+n}^{t+n-1} \frac{1}{i}$$
$$+ \frac{B_2}{2} x^{-2}\Big|_t^{t+n} + \frac{B_4}{4} x^{-4}\Big|_t^{t+n} + \dots .$$

So

$$\ln n - \sum_{1}^{n} \frac{1}{t+i} \mathrel{\dot\sim} \ln\frac{n}{t+n} - \frac{1}{t+n} + \ln t + \frac{1}{2t} + \frac{1}{2(t+n)}$$
$$+ \frac{B_2}{2} x^{-2}\Big|_t^{t+n} + \frac{B_4}{4} x^{-4}\Big|_t^{t+n} + \dots,$$

thus

$$\text{(3.17)} \qquad \lim\left(\ln n - \sum_{1}^{n} \frac{1}{t+i}\right) \mathrel{\dot\sim} \ln t + \frac{1}{2t} - \frac{B_2}{2t^2} - \frac{B_4}{4t^4} - \frac{B_6}{6t^6} - \dots,$$

i.e., the RHS of (3.17) is the asymptotic expansion of the derivative of $\ln\Gamma(t+1)$. But the asymptotic expansion of an integral can be obtained by integrating the asymptotic expansion of its integrand term by term; so

$$\text{(3.18)} \quad \ln\Gamma(t+1) \mathrel{\dot\sim} C + (t+\tfrac{1}{2})\ln t - t + \frac{B_2}{1\cdot 2t} + \frac{B_4}{3\cdot 4t^3} + \frac{B_6}{5\cdot 6t^5} + \dots .$$

The constant $C$ is obtained from the relation

$$\text{(3.19)} \quad \ln\Gamma(t+1) + \ln\Gamma(t+\tfrac{1}{2}) = -2t\ln 2 + \tfrac{1}{2}\ln\pi + \ln\Gamma(2t+1),$$

which yields $C = \frac{1}{2}\ln 2\pi$. We have derived *Stirling's formula*:

$$\text{(3.20)} \quad \ln\Gamma(t+1) \mathrel{\dot\sim} \tfrac{1}{2}\ln 2\pi + (t+\tfrac{1}{2})\ln t - t + \frac{B_2}{1\cdot 2t} + \frac{B_4}{3\cdot 4t^3} + \frac{B_6}{5\cdot 6t^5} + \dots .$$

The error in applying this asymptotic expansion is numerically less than the first term that is omitted.

CHAPTER 6

# ASYMPTOTIC EXPANSIONS – THE FORMAL CONCEPT

## 1. Asymptotic sequences; asymptotic expansions

The approach of this chapter is based on the work of van der Corput who wrote many papers devoted to asymptotics (e.g. see van der Corput [1955/56]). The subject, in this form, goes back to Poincaré (see Poincaré [1886]).

In Sections 5.1 and 5.3 we have presented several asymptotic expansions, which we now summarize:

$$(1)\ \mathrm{ei}(t) \sim \mathrm{e}^{-t}\left[\frac{1}{t} - \frac{1}{t^2} + \dots + (-1)^n \frac{n!}{t^{n+1}} + \dots\right],$$

$$(2)\ E_i(t) \sim \frac{1}{t} - \frac{1}{t^2} + \dots + (-1)^n \frac{n!}{t^{n+1}} + \dots,$$

$$(3)\ \ln t - \sum_1^t \frac{1}{i} \sim -\frac{1}{2t} - \gamma + \frac{B_2}{2t^2} + \frac{B_4}{4t^4} + \dots,$$

$$(4)\ \ln\Gamma(t+1) + t - (t+\tfrac{1}{2})\ln t \sim \tfrac{1}{2}\ln 2\pi + \frac{B_2}{1\cdot 2t} + \frac{B_4}{3\cdot 4t^3} + \dots,$$

$$(5)\ \sum_{t+1}^{2t} \frac{1}{i} \sim \ln 2 - \frac{1}{4t} + B_2\frac{2^2-1}{2\cdot 2^2}t^{-2} + B_4\frac{2^4-1}{4\cdot 2^4}t^{-4} + \dots,$$

and

$$(6)\ \sum_{t+1}^{\infty} i^{-p} \sim \frac{t^{1-p}}{p-1} - \frac{t^{-p}}{2} + \frac{B_2}{2}\binom{p}{1}t^{-p-1} + \frac{B_4}{4}\binom{p+2}{3}t^{-p-3} + \frac{B_6}{6}\binom{p+4}{5}t^{-p-5} + \dots,$$

provided $p > 1$.

The RHS of each of the above expansions has the form $\Sigma\, a_i\phi_i(t)$, and each LHS has the form $f(t)$. Moreover, each expression $\Sigma\, a_i\phi_i(t)$ generates approximations to $f(t)$ by forming a partial sum $\Sigma_0^n a_i\phi_i(t)$ of $\Sigma\, a_i\phi_i(t)$. The absolute value of the error in this approximation is less than $|a_{n+1}\phi_{n+1}(t)|$, the absolute value of the next term of $\Sigma\, a_i\phi_i$.

Examining these expansions, we see that they share the following properties:

(i) For each $n \in N$, $\phi_n$ has no zero in $R$.

(ii) $\lim_\infty \phi_{n+1}/\phi_n = 0$ for each $n \in N$.

(iii) For each $t \in \mathrm{dom} f$, $|f(t) - \Sigma_0^n a_i\phi_i(t)| < |a_{n+1}\phi_{n+1}(t)|$, where $n \in N$.

(iv) $f - \Sigma_0^n a_i\phi_i = \mathrm{o}(\phi_n)$ for each $n \in N$; i.e., $\lim_\infty (f - \Sigma_0^n a_i\phi_i)/\phi_n = 0$ for each $n \in N$.

To verify (iv), notice that for each $n \in N$,

$$\left|\frac{f(t) - \Sigma_0^n a_i\phi_i(t)}{\phi_n(t)}\right| < \left|\frac{a_{n+1}\phi_{n+1}(t)}{\phi_n(t)}\right|$$

by (iii). But

$$\lim_\infty \frac{a_{n+1}\phi_{n+1}(t)}{\phi_n(t)} = a_{n+1} \lim_\infty \frac{\phi_{n+1}(t)}{\phi_n(t)} = 0$$

by (ii). So

$$\lim_\infty \frac{f(t) - \Sigma_0^n a_i\phi_i(t)}{\phi_n(t)} = 0.$$

We now present our formal definitions; first we define the concept of an asymptotic sequence, and then we define the notion of an asymptotic expansion of a function. A sequence of standard functions $\phi_0, \phi_1, \phi_2, \ldots$ is said to be *asymptotic* provided that:

(a) there is a neighbourhood of $\infty$ in which no term of the sequence has a zero;

(b) for each $n \in N$, $\phi_{n+1} = \mathrm{o}(\phi_n)$; i.e., $\lim_\infty \phi_{n+1}/\phi_n = 0$ for each $n \in N$.

Notice that (b) can be expressed as follows: $\phi_{n+1}(\kappa)/\phi_n(\kappa) \simeq 0$ for each $n \in N$ and for each positive, infinite $\kappa$.

For example, the sequence $(x^{-i})$ and $(\mathrm{e}^{-ix})$ are asymptotic. More generally, let $(\nu_i)$ be a strictly increasing sequence of standard numbers; then $(x^{-\nu_i})$ and $(\mathrm{e}^{-\nu_i})$ are asymptotic sequences.

Turning to our second concept, let $(\phi_i)$ be any asymptotic sequence, let $(a_i)$ be a sequence of standard numbers, and let $f$ be a standard function whose domain contains a neighbourhood of $\infty$. Then the formal expression $\Sigma\, a_i\phi_i$, which is also denoted by writing $a_0\phi_0 + a_1\phi_1 + a_2\phi_2 + \ldots$, is called an *asymptotic expansion for* $f$ provided that for each $n \in N$,

$$f - \sum_0^n a_i \phi_i = o(\phi_n),$$

i.e.,

$$\lim_\infty (f - \sum_0^n a_i \phi_i)/\phi_n = 0.$$

In this case, we write $f \sim \Sigma\, a_i\phi_i$ or $f \sim a_0\phi_0 + a_1\phi_1 + a_2\phi_2 + \ldots$.

At the beginning of this section we have presented asymptotic expansions for the functions ei, $E_i$, $\ln t - \Sigma_1^t 1/i$, $\ln \Gamma(t+1) + t - (t+\frac{1}{2})\ln t$, $\Sigma_{t+1}^{2t} 1/i$, and $\Sigma_{t+1}^\infty i^{-p}$, where $p > 1$.

When we assert that $f \sim \Sigma\, a_i\phi_i$, we imply not only that $f - \Sigma_0^n a_i\phi_i = o(\phi_n)$, but also that $f$ is a function, $(a_i)$ is a sequence of standard numbers, and $(\phi_i)$ is an asymptotic sequence.

Here are a few facts about asymptotic expansions. First we show that the coefficients $a_i$ are determined by $f$, once the asymptotic sequence $(\phi_i)$ is given.

1.1. LEMMA. *Let* $f \sim \Sigma\, a_i\phi_i$. *Then* $a_0 = \lim_\infty f/\phi_0$ *and*

$$a_n = \lim_\infty \frac{f - \Sigma_0^{n-1} a_i\phi_i}{\phi_n}$$

*for each* $n > 0$.

*Proof.* By assumption, for each $n \in N$,

$$\lim_\infty \frac{f - \Sigma_0^n a_i\phi_i}{\phi_n} = 0. \tag{1.2}$$

Taking $n = 0$ in (1.2) yields

$$\lim_\infty \frac{f - a_0\phi_0}{\phi_0} = 0,$$

i.e. $\lim_\infty f/\phi_0 = a_0$. For $n > 0$ we obtain from (1.2),

$$\lim_\infty \frac{f - \Sigma_0^{n-1} a_i\phi_i - a_n\phi_n}{\phi_n} = 0,$$

thus

$$\lim_{\infty} \frac{f - \Sigma_0^{n-1} a_i\phi_i}{\phi_n} = a_n.$$

This completes our proof. □

We have not proven that each function $f$ has a *unique* asymptotic expansion; merely that there is at most one asymptotic expansion for $f$ of the form $\Sigma a_i\phi_i$, where the asymptotic sequence $(\phi_i)$ is fixed in advance. To illustrate this point, notice that the function $1/(x^2 - 1)$ has several asymptotic expansions; e.g.

$$\frac{1}{x^2-1} \sim \sum x^{-2i-2}, \qquad \frac{1}{x^2-1} \sim \sum (x^2+1)\, x^{-4i-4}.$$

However, these asymptotic expansions reduce to the same asymptotic power series (see Section 6.2), and for this reason may be regarded as essentially the same. To better illustrate our point that a function may possess two asymptotic expansions, consider the function $\ln(1 + 1/x)$. Let $t$ be any positive standard integer; from (5.3.5),

$$\ln\left(1+\frac{1}{t}\right) \sim \frac{1}{2}\left(\frac{1}{t}+\frac{1}{t+1}\right) + \frac{B_2}{2}\left(\frac{1}{(t+1)^2}-\frac{1}{t^2}\right) + \frac{B_4}{4}\left(\frac{1}{(t+1)^4}-\frac{1}{t^4}\right) + \ldots,$$

and by Example 2.8,

$$\ln\left(1+\frac{1}{t}\right) \sim t^{-1} - \tfrac{1}{2}t^{-2} + \tfrac{1}{3}t^{-3} - \ldots .$$

Moreover, several functions can have the same asymptotic expansion. Let $g \sim \Sigma\, 0\phi_i$, and let $f \sim \Sigma\, a_i\phi_i$. Then for each $n \in N$,

$$\lim_{\infty} \frac{f+g-\Sigma_0^n a_i\phi_i}{\phi_n} = \lim_{\infty} \frac{f-\Sigma_0^n a_i\phi_i}{\phi_n} + \lim_{\infty} \frac{g}{\phi_n} = 0$$

by assumption. So $f + g \sim \Sigma a_i\phi_i$; thus $\Sigma a_i\phi_i$ is an asymptotic expansion for both $f$ and $f + g$. Let us illustrate this point; now,

$$\frac{1}{1+x} \sim \sum (-1)^i x^{-i-1}, \qquad e^{-x} \sim \sum 0x^{-i-1}.$$

So

$$\frac{1}{1+x} + e^{-x} \sim \sum (-1)^i x^{-i-1}.$$

Thus $\Sigma (-1)^i x^{-i-1}$ is an asymptotic expansion for both $1/(1+x)$ and $1/(1+x) + e^{-x}$.

Next we present a necessary and sufficient condition that a given expression $\Sigma a_i\phi_i$ (which is also called an *asymptotic series*) be an asymptotic expansion for a given function $f$.

1.3. CRITERION FOR ASYMPTOTIC EXPANSIONS. *$f \sim \Sigma a_i\phi_i$ iff $(f - \Sigma_0^{n-1} a_i\phi_i)/\phi_n$ is bounded on some neighbourhood of ∞ whenever $n > 0$.*

*Proof.* It may be helpful to spell out our criterion: corresponding to each positive standard natural number $n$ there are standard numbers $B$ and $q$ such that for each standard number $s > q$,

$$\left| \frac{f(s) - \Sigma_0^{n-1} a_i\phi_i(s)}{\phi_n(s)} \right| < B. \tag{1.4}$$

There are two parts to our proof.

(i) Assume that $f \sim \Sigma a_i\phi_i$. By Lemma 1.1, for each $n > 0$,

$$\lim_\infty \frac{f - \Sigma_0^{n-1} a_i\phi_i}{\phi_n} = a_n.$$

In particular, then, $(f - \Sigma_0^{n-1} a_i\phi_i)/\phi_n$ is bounded in some neighbourhood of ∞ by $1 + |a_n|$. This establishes (1.4).

(ii) Assume (1.4). For each $n \in N$,

$$\frac{f - \Sigma_0^n a_i\phi_i}{\phi_n} = \left(\frac{f - \Sigma_0^n a_i\phi_i}{\phi_{n+1}}\right)\frac{\phi_{n+1}}{\phi_n}. \tag{1.5}$$

Notice that $\lim_\infty \phi_{n+1}/\phi_n = 0$ since $(\phi_i)$ is an asymptotic sequence; also, the first factor on the RHS of (1.5) is bounded in some neighbourhood of ∞ by assumption. So, for each $n \in N$,

$$\lim_\infty \frac{f - \Sigma_0^n a_i\phi_i}{\phi_n} = 0.$$

Thus $f \sim \Sigma a_i\phi_i$. This completes our proof. □

Asymptotic expansions can be added; i.e., if $f \sim \Sigma a_i\phi_i$ and $g \sim \Sigma b_i\phi_i$, then

$$f+g \sim \Sigma (a_i + b_i)\phi_i.$$

Also, if $f \sim \Sigma a_i\phi_i$ and $c \in R$, then

$$cf \sim \Sigma ca_i\phi_i.$$

Generally, we cannot multiply asymptotic expansions; however, we can multiply asymptotic expansions of the form $\Sigma a_i x^{-i}$, and we can divide asymptotic expansions of this form provided that the first coefficient of the denominator is nonzero. By this we mean the following. Let $f \sim \Sigma a_i x^{-i}$, and let $g \sim \Sigma b_i x^{-i}$; then

$$f \cdot g \sim \Sigma c_i x^{-i},$$

where, for each $i$,

$$c_i = a_0 b_i + a_1 b_{i-1} + \ldots + a_i b_0.$$

Also, if $a_0 \neq 0$, then

$$\frac{1}{f} \sim \sum d_i x^{-i},$$

where the coefficients $d_i$ are computed by applying Lemma 1.1; so

$$d_0 = \lim_\infty \frac{1}{f} = \frac{1}{a_0},$$

$$d_1 = \lim_\infty x\left(\frac{1}{f} - d_0\right) = -\frac{a_1}{a_0^2},$$

$$d_2 = \lim_\infty x^2\left(\frac{1}{f} - d_0 - \frac{d_1}{x}\right) = \frac{a_1^2 - a_0 a_2}{a_0^3},$$

and so on. More simply, if $i > 0$ then

$$d_i = -\frac{a_1 d_{i-1} + \ldots + a_i d_0}{a_0}.$$

Let $\psi$ be a function that has no zeros in some neighbourhood of $\infty$. Clearly, $(\psi \cdot \phi_i)$ is an asymptotic sequence iff $(\phi_i)$ is an asymptotic sequence. Moreover,

$$\lim_\infty \frac{f - \psi\, \Sigma_0^n a_i\phi_i}{\psi\phi_n} = \lim_\infty \frac{f/\psi - \Sigma_0^n a_i\phi_i}{\phi_n},$$

so $f \sim \Sigma a_i\, \psi \cdot \phi_i$ iff $f/\psi \sim \Sigma a_i\phi_i$.

The proof of our next lemma utilizes our Criterion 1.3.

1.6. LEMMA. *Let $f$ be a continuous function such that $f \sim \Sigma a_i x^{-i}$ and* $\mathrm{dom} f = R$. *Then $\int_t^\infty (f - a_0 - a_1/x)$ converges provided that $t > 0$.*

*Proof.* By our Criterion, with $n = 1$, $x^2(f - a_0 - a_1/x)$ is bounded on a neighbourhood of $\infty$. Therefore, there are standard numbers $B$ and $q$ such that

$$\forall s\,[s > q \rightarrow |f(s) - a_0 - a_1/s| < B/s^2]. \tag{1.7}$$

Moreover,

$$\int_t^\infty \left(f - a_0 - \frac{a_1}{x}\right) = \int_t^q \left(f - a_0 - \frac{a_1}{x}\right) + \int_q^\infty \left(f - a_0 - \frac{a_1}{x}\right). \tag{1.8}$$

The first integral on the RHS converges since its integrand is continuous on the open interval $(t, q)$. The improper integral on the RHS converges by the Comparison Test for improper integrals; indeed, (1.7) asserts that its integrand is bounded by $B/x^2$, and of course $\int_q^\infty B/x^2$ converges. □

Later we shall require the following corollary.

1.9. COROLLARY. *Let $f \sim \Sigma a_i x^{-i}$, let $f$ be continuous on $R$, and let $\int_t^\infty f$ converge for some $t > 0$. Then $a_0 = a_1 = 0$.*

*Proof.* If the improper integrals $\int_t^\infty f$ and $\int_t^\infty (f+g)$ both converge, then $\int_t^\infty g$ converges. But $\int_t^\infty f$ converges by assumption, and $\int_t^\infty (f - a_0 - a_1/x)$ converges by Lemma 1.6; so $\int_t^\infty (a_0 + a_1/x)$ converges. It follows that $a_0 = a_1 = 0$. □

The following fact is useful.

1.10. LEMMA. $0 \sim \Sigma a_i \phi_i$ *iff each* $a_i = 0$.

*Proof.* Clearly, $0 \sim \Sigma\, 0\phi_i$. Next assume that $0 \sim \Sigma a_i \phi_i$; we shall show that $a_i = 0$ for each $i \in N$. By Lemma 1.1,

$$a_0 = \lim_\infty \frac{0}{\phi_0} = 0,$$

$$a_1 = \lim_\infty \frac{a_0 \phi_0}{\phi_1} = \lim_\infty \frac{0}{\phi_1} = 0,$$

$$a_2 = \lim_{\infty} \frac{a_0\phi_0 + a_1\phi_1}{\phi_2} = \lim_{\infty} \frac{0}{\phi_2} = 0,$$

and in general,

$$a_{n+1} = \lim_{\infty} \frac{\Sigma_0^n a_i\phi_i}{\phi_{n+1}} = \lim_{\infty} \frac{0}{\phi_{n+1}} = 0,$$

provided that $a_i = 0$ for each $i \leqslant n$. We conclude by mathematical induction that $a_i = 0$ for each $i \in N$. □

## 2. Asymptotic power series

Five of the six asymptotic expansions which were developed in Sections 5.1 and 5.3, and listed in Section 1, have the form $\Sigma a_i x^{-i}$, where some of the coefficients may be zero; only the expansion of the *incomplete factorial* function ei does not have this form. Expressions that have the form $\Sigma a_i x^{-i}$, where $a_i \in R$ for each $i \in N$, are called *asymptotic power series*. In Section 6.1 we have mentioned some basic properties of asymptotic power series. Here we shall prove that asymptotic expansions of this special kind can be integrated and differentiated.

2.1. LEMMA. *Let $f \sim \Sigma a_i x^{-i}$, let $f$ be continuous on $R$, and let $F$ be the function such that*

$$F(t) = \int_t^{\infty} \left(f - a_0 - \frac{a_1}{x}\right) \quad \textit{for } t > 0.$$

*Then*

$$F \sim \frac{a_2}{x} + \frac{a_3}{2x^2} + \ldots + \frac{a_{i+2}}{(i+1)x^{i+1}} + \ldots .$$

*Proof.* By Lemma 1.6, $t \in \operatorname{dom} F$ if $t > 0$. We shall use our Criterion for Asymptotic Expansions 1.3. By this Criterion, each function $x^{n+1}(f - \Sigma_0^n a_i x^{-i})$ is bounded in a neighbourhood of $\infty$ for each $n \in N$. It follows that if $t$ is large enough,

$$t^n \int_t^{\infty} \left(f - \sum_0^n a_i x^{-i}\right)$$

is bounded, i.e.,

$$t^n\left[F(t)-\int_t^\infty \sum_2^n a_i x^{-i}\right]$$

is bounded. But

$$\int_t^\infty (a_2x^{-2}+\ldots+a_nx^{-n}) = \frac{a_2}{t}+\frac{a_3}{2t^2}+\ldots+\frac{a_n}{(n-1)t^{n-1}}$$

so

$$x^n\left[F-\left(\frac{a_2}{x}+\frac{a_3}{2x^2}+\ldots+\frac{a_n}{(n-1)x^{n-1}}\right)\right]$$

is bounded in a neighbourhood of $\infty$. Thus, by our Criterion,

$$F\sim\sum\frac{a_{i+2}}{(i+1)x^{i+1}}.$$

This completes our proof. □

We have proved that if $f-a_0-a_1/x\sim\Sigma\, a_{i+2}x^{-(i+2)}$, then

$$\text{(2.2)}\qquad \int_x^\infty f(t)-a_0-\frac{a_1}{t}\ \mathrm{d}t\sim\sum b_{i+1}x^{-(i+1)},$$

where

$$b_{i+1}x^{-(i+1)}=\int_x^\infty a_{i+2}\,t^{-(i+2)}\,\mathrm{d}t$$

for each $i\in N$. By Corollary 1.9, the LHS of (2.2) simplifies to $\int_x^\infty f$ in case this improper integral converges. In this sense, we have proved that the asymptotic power series expansion of the integral of a function $f$ is obtained from the asymptotic power series expansion of $f$ by integrating term by term. Remember, this is based on the assumption that the improper integral of $f$ converges.

Next we shall prove that the asymptotic power series expansion of the derivative of a function $f$ (if the expansion exists) is obtained from the asymptotic power series expansion of $f$ by differentiating term by term.

2.3. LEMMA. *Let* $f \sim \Sigma\, a_i x^{-i}$, *let* $f'$ *be continuous on* $R$ *and have an asymptotic power series expansion. Then*

$$f' \sim \Sigma -(i+1)\, a_{i+1} x^{-(i+2)}.$$

*Proof*. Let $f' \sim \Sigma\, b_i x^{-i}$, and let $F$ be the function such that

$$F(t) = \int_t^\infty \left(f' - b_0 - \frac{b_1}{x}\right) \quad \text{for } t > 0.$$

By Lemma 1.6, $t \in \operatorname{dom} F$ if $t > 0$. Now $\int_t^\infty f'$ converges since

$$\int_t^\infty f' = f\Big|_t^\infty = a_0 - f(t)$$

(recall that $\lim_\infty f = a_0$). Thus, by Corollary 1.9, $b_0 = b_1 = 0$; so

$$F(t) = \int_t^\infty f' = a_0 - f(t)$$

for each $t > 0$, i.e., $F = a_0 - f$. By Lemma 2.1,

$$a_0 - f \sim \frac{b_2}{x} + \frac{b_3}{2x^2} + \ldots + \frac{b_{i+2}}{(i+1)\, x^{i+1}} + \ldots,$$

so

$$f \sim a_0 - \frac{b_2}{x} - \frac{b_3}{2x^2} - \ldots - \frac{b_{i+2}}{(i+1)\, x^{i+1}} - \ldots .$$

By assumption,

$$f \sim a_0 + a_1/x + \ldots + a_i/x^i + \ldots .$$

Therefore

$$-\frac{b_{i+2}}{i+1} = a_{i+1}$$

for each $i \in N$, since $f$ has just one asymptotic power series expansion. Thus $f' \sim \Sigma -(i+1)\, a_{i+1} x^{-(i+2)}$. □

We now utilize the lemmas of this section to derive some asymptotic power series expansions.

2.4. EXAMPLE. Let

$$E = \int_x^\infty e^{-t^2}\, dt;$$

it is known that $e^{x^2} \cdot E$ has an asymptotic power series expansion, say $\Sigma a_i x^{-i}$, which we shall now derive. Now

$$(e^{x^2} \cdot E)' = 2x\, e^{x^2} \cdot E - 1.$$

Thus, by Lemma 2.3,

$$2x\, e^{x^2} \cdot E - 1 \sim -a_1 x^{-2} - 2a_2 x^{-3} - 3a_3 x^{-4} - \ldots$$

so

$$e^{x^2} \cdot E \sim \frac{1}{2x}\,[1 - a_1 x^{-2} - 2a_2 x^{-3} - 3a_3 x^{-4} - \ldots].$$

Therefore $a_0 = 0$, $a_1 = \frac{1}{2}$, $a_2 = 0$, $a_3 = -\frac{1}{2}a_1$, and in general $a_{n+2} = -\frac{1}{2}na_n$ for each $n \in N$. Thus

$$\sum a_i x^{-i} = \frac{1}{2x}\left(1 - \frac{1}{2x^2} + \frac{3}{(2x^2)^2} - \frac{3 \cdot 5}{(2x^2)^3} + \frac{3 \cdot 5 \cdot 7}{(2x^2)^4} - \ldots\right)$$

i.e.

$$E \sim \frac{e^{-x^2}}{2x}\left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \ldots\right).$$

In Example 5.1.1, we worked out an asymptotic expansion for the incomplete factorial function $\mathrm{ei} = \int_x^\infty (e^{-t}/t)\, dt$. Our next example derives this expansion by applying Lemma 2.3.

2.5. EXAMPLE. Let $e^x \cdot \mathrm{ei} \sim \Sigma a_i x^{-i}$. Then by Lemma 2.3,

$$e^x \cdot \mathrm{ei} - x^{-1} \sim -a_1 x^{-2} - 2a_2 x^{-3} - 3a_3 x^{-4} - \ldots,$$

so

$$e^x \cdot \mathrm{ei} \sim x^{-1} - a_1 x^{-2} - 2a_2 x^{-3} - 3a_3 x^{-4} - \ldots.$$

Therefore $a_0 = 0$, $a_1 = 1$, $a_2 = -a_1$, $a_3 = -2a_2$, and in general $a_{n+1} = -na_n$ for each $n \in N$. Thus

$$e^x \cdot \mathrm{ei} \sim x^{-1} - x^{-2} + 2x^{-3} - 3!\, x^{-4} + \ldots + (-1)^i i!\, x^{-(i+1)} + \ldots,$$

so

$$\mathrm{ei} \sim e^{-x} \sum (-1)^i i!\, x^{-(i+1)}.$$

2.6. EXAMPLE. Next we shall derive the asymptotic power series expansion of the function arctan. Since $\arctan' = 1/(1+x^2)$ we begin by observing that

$$\frac{1}{1+x^2} \sim x^{-2} - x^{-4} + x^{-6} - \dots . \tag{2.7}$$

Now $\int_x^\infty 1/(1+t^2)\,\mathrm{d}t$ converges; so by Lemma 2.1,

$$\int_x^\infty \frac{1}{1+t^2}\,\mathrm{d}t \sim x^{-1} - \tfrac{1}{3}x^{-3} + \tfrac{1}{5}x^{-5} - \dots,$$

thus

$$\arctan \sim \tfrac{1}{2}\pi - x^{-1} + \tfrac{1}{3}x^{-3} - \tfrac{1}{5}x^{-5} + \dots .$$

Moreover, applying Lemma 2.3 to (2.7) yields the asymptotic power series expansion of $1/(1+x^2)^2$ more easily than by squaring the RHS of (2.7). We obtain

$$-\frac{2x}{(1+x^2)^2} \sim -2x^{-3} + 4x^{-5} - 6x^{-7} + \dots,$$

so

$$\frac{1}{(1+x^2)^2} \sim x^{-4} - 2x^{-6} + 3x^{-8} - \dots + (-1)^i(i+1)x^{-2(i+2)} + \dots .$$

2.8. EXAMPLE. We shall develop the asymptotic power series expansion for $\ln(1+1/x)$. Now

$$\mathrm{D}\ln\left[1+\frac{1}{x}\right] = -\frac{1}{x(1+x)}$$

and

$$-\frac{1}{x(1+x)} \sim -x^{-2} + x^{-3} - x^{-4} + \dots .$$

Notice that

$$\int_t^\infty -\frac{1}{x(1+x)} = -\ln\left(1+\frac{1}{t}\right)$$

if $t > 0$; thus, by Lemma 2.1,

$$\ln\left(1+\frac{1}{x}\right) \sim x^{-1} - \tfrac{1}{2}x^{-2} + \tfrac{1}{3}x^{-3} - \dots .$$

We can use the technique of this section to prove that certain functions do not possess asymptotic power series expansions.

2.9. EXAMPLE. We shall prove that $x^{-1/2}$ does not have an asymptotic power series expansion. Assume that $x^{-1/2} \sim \Sigma a_i x^{-i}$; by Lemma 2.3,

$$x^{-3/2} \sim 2(a_1 x^{-2} + 2a_2 x^{-3} + 3a_3 x^{-4} + \ldots).$$

But the product of two asymptotic power series expansions is the asymptotic power series expansion of the product of the functions involved; so

$$x^{-2} \sim 2a_0 a_1 x^{-2} + 2(a_0 a_2 + a_1^2) x^{-3} + \ldots .$$

Clearly $x^{-2} \sim x^{-2}$; so $2a_0 a_1 = 1$, thus $a_0 \neq 0$. By Lemma 1.1,

$$a_0 = \lim_{\infty} x^{-1/2} = 0.$$

This contradiction proves that $x^{-1/2}$ has no asymptotic power series expansion.

2.10. EXAMPLE. We shall determine asymptotic expansions of solutions of the differential equation

$$\frac{1}{x} f'' + x f' + f = 0. \tag{2.11}$$

Let $f \sim \Sigma c_i x^{\sigma - i}$, where $c_0 \neq 0$. Then

$$x f' \sim \Sigma (\sigma - i) c_i x^{\sigma - i},$$

$$\frac{1}{x} f'' \sim \Sigma (\sigma - i)(\sigma - i - 1) c_i x^{\sigma - i - 3}.$$

Thus

$$0 \sim \Sigma (\sigma - i + 1) c_i x^{\sigma - i} + \Sigma (\sigma - i)(\sigma - i - 1) c_i x^{\sigma - i - 3}. \tag{2.12}$$

But $0 \sim \Sigma a_i x^{-\nu_i}$ iff each $a_i = 0$; thus, from (2.12), $(\sigma + 1) c_0 = 0$, so $\sigma = -1$. From (2.12),

$$0 \sim \Sigma -i c_i x^{-i-1} + \Sigma (i + 1)(i + 2) c_i x^{-i-4}; \tag{2.13}$$

therefore, for each $i$,

$$(i + 1)(i + 2) c_i = (i + 3) c_{i+3},$$

so

$$c_{i+3} = \frac{(i + 1)(i + 2)}{i + 3} c_i,$$

thus

$$c_{3i} = \frac{(3i)!}{(i!\,3^i)^2}\, c_0, \qquad c_{3i+1} = c_{3i+2} = 0$$

for each $i$. We conclude that

$$f \sim c_0(x^{-1} + \tfrac{2}{3}x^{-4} + \tfrac{20}{9}x^{-7} + \tfrac{1120}{81}x^{-10} + ...);$$

i.e.

$$f \sim \sum \frac{(3i)!}{(i!\,3^i)^2}\, c_0 x^{-3i-1}.$$

Let $f$ be the solution of (2.11) for which $\lim_\infty xf = 1$, i.e., $c_0 = 1$. Then

$$f \sim x^{-1} + \tfrac{2}{3}x^{-4} + \tfrac{20}{9}x^{-7} + \tfrac{1120}{81}x^{-10} + \ldots.$$

For example,

$$f(10) \doteq .1 + .000067 + .0000002 = .1000672,$$

which is correct as far as it goes.

The theory of this chapter extends to functions of a complex variable. In the complex plane, $z$ can approach $\infty$ along many paths; whereas in the real plane essentially just one path, the positive $x$-axis, is available. It turns out that an asymptotic expansion of an analytic function is usually *not* valid over all paths that approach $\infty$, but is valid if the paths are restricted to a certain sector of the complex plane, a sector which depends on the function involved. For this reason, we normally speak of an asymptotic expansion over a *sector*, and define this concept relative to a given sector. So we define $f \sim \Sigma\, a_i\phi_i$, where $(\phi_i)$ is an asymptotic sequence, provided that for each $n \in N$ and for each $\epsilon > 0$ there is a real number $B$ such that

$$\text{(2.14)} \qquad \frac{|f(z) - \Sigma_0^n a_i\phi_i|}{|\phi_n(z)|} < \epsilon$$

whenever $|z| > B$ and $\arg z$ is in the interval that characterizes the given sector of the complex plane. In short, (2.14) must hold for each member of the sector, say $z$, such that $|z| > B$. We say that (2.14) holds *uniformly* over the sector involved.

Generally, our results about asymptotic expansions of a function of a real variable carry over to functions of a complex variable with only minor qualifications. In particular, the uniqueness of the coefficients of an asymptotic expansion of $f$, relative to a given asymptotic sequence, is valid only over a sector of the complex plane. An analytic function may have different asymptotic expansions, each involving the same asymptotic sequence, over different sectors of the complex plane; this is known as *Stokes' phenomenon*.

On the other hand, if an analytic function $f$ possesses an asymptotic power series expansion $\Sigma a_i z^{-i}$ that is valid without any restriction on $\arg z$, then $f(z) = \Sigma_0^\infty a_i z^{-i}$ if $|z|$ is large enough (compare Copson [1965]).

2.15. LEMMA. *Let $f$ be analytic in a neighbourhood of $\infty$ except possibly at $z = \infty$, and let $f \sim \Sigma a_i z^{-i}$ with no restriction on* $\arg z$. *Then there is a real number $q$ such that $f(z) = \Sigma_0^\infty a_i z^{-i}$ if $|z| > q$.*

*Proof.* By assumption, $\lim_\infty f = a_0$; so $f$ is bounded in a neighbourhood of $\infty$ with $\infty$ deleted; i.e., there are real numbers $B$ and $q$ such that $|f(z)| < B$ if $|z| > q$; also, we can assume that $f$ is analytic in this neighbourhood of $\infty$ except possibly at $z = \infty$. Thus $f$ has a Laurent series expansion, say $\Sigma_{-\infty}^\infty b_n z^n$, which is valid for $|z| > q$; i.e., $f(z) = \Sigma_{-\infty}^\infty b_n z^n$ if $|z| > q$. Moreover, for each $n > 0$,

$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{f}{z^{n+1}},$$

where $\gamma$ is *any* circle with radius $r > q$ and center at the origin. Clearly,

$$|b_n| < \frac{1}{2\pi} \frac{B}{r^{n+1}} 2\pi r = \frac{B}{r^n}.$$

Since we are free to assign any value to $r$, subject only to the restriction $r > q$, it follows that $b_n = 0$ if $n > 0$. Therefore

$$f(z) = \sum_0^\infty b_{-n} z^{-n} \quad \text{if } |z| > q.$$

By Lemma 1.1,

$$a_0 = \lim_\infty f = b_0,$$

$$a_1 = \lim_\infty z(f - a_0) = b_{-1},$$

$$a_2 = \lim_\infty z^2(f - a_0 - a_1 z^{-1}) = b_{-2},$$

and in general

$$a_{n+1} = \lim_{\infty} z^{n+1}\left(f - \sum_0^n a_i z^{-i}\right) = \lim_{\infty} z^{n+1}\left(f - \sum_0^n b_{-i} z^{-i}\right) = b_{-(n+1)}.$$

So $f(z) = \Sigma_0^\infty a_i z^{-i}$ if $|z| > q$. □

2.16. EXAMPLE. We shall determine asymptotic expansions of solutions of the differential equation

$$(2.17) \qquad f'' + \left(1 - \frac{1}{x^2}\right) f = 0.$$

We anticipate that $f$ is a product of the form $e^{ax} g$; substituting $e^{ax} g$ for $f$ in (2.17) yields

$$(2.18) \qquad g'' + 2ag' + \left(a^2 + 1 - \frac{1}{x^2}\right) g = 0.$$

The idea is to choose $a$ so that we can solve (2.18) asymptotically for the unknown function $g$. To this purpose, let $g \sim \Sigma c_j x^{\sigma - j}$, where $c_0 \neq 0$. Now

$$g' \sim \Sigma (\sigma - j) c_j x^{\sigma - j - 1},$$
$$g'' \sim \Sigma (\sigma - j)(\sigma - j - 1) c_j x^{\sigma - j - 2};$$

so (2.18) yields

$$(2.19) \qquad 0 \sim (a^2 + 1) \Sigma c_j x^{\sigma - j} + 2a \Sigma (\sigma - j) c_j x^{\sigma - j - 1} + \Sigma [(\sigma - j)(\sigma - j - 1) - 1] c_j x^{\sigma - j - 2}.$$

Therefore $(a^2 + 1) c_0 = 0$, so $a^2 + 1 = 0$, $a = \sqrt{-1} = \mathrm{i}$. From (2.19),

$$(2.20) \qquad 0 \sim 2\mathrm{i} \, \Sigma (\sigma - j) c_j x^{\sigma - j - 1} + \Sigma [(\sigma - j)(\sigma - j - 1) - 1] c_j x^{\sigma - j - 2},$$

so $\sigma c_0 = 0$, thus $\sigma = 0$. We obtain

$$(2.21) \qquad 0 \sim -2\mathrm{i} \, \Sigma j c_j x^{-j-1} + \Sigma [j(j+1) - 1] c_j x^{-j-2}.$$

Therefore

$$2\mathrm{i}(j+1) c_{j+1} = [j(j+1) - 1] c_j$$

for each $j$; thus

$$c_{j+1} = \frac{j^2 + j - 1}{2\mathrm{i}(j+1)} c_j$$

for each $j$, and $g \sim \Sigma c_j x^{-j}$. We have found asymptotic expansions of solutions of (2.18) in terms of the parameter $c_0$. Returning to the differential equation (2.17), we conclude that $e^{-ix} f \sim \Sigma c_j x^{-j}$, where

$$c_{j+1} = \frac{j^2 + j - 1}{2i(j+1)} c_j$$

for each $j$; i.e.,

$$e^{-ix} f \sim c_0 \left(1 + \tfrac{1}{2} ix^{-1} + \tfrac{1}{8} x^{-2} - \tfrac{5}{48} ix^{-3} - \tfrac{55}{384} x^{-4} + \ldots\right).$$

We draw attention to the fact that by identifying $t$ with $1/x$, we can associate with each asymptotic power series $\Sigma a_i x^{-i}$ a field element of the nonarchimedean field $\mathcal{L}$, namely $\Sigma_N a_i t^i$ (see Section 1.7). Identifying $t$ with $1/x$ is justified on the grounds that in asymptotics we are concerned with evaluating the functions $x^i$ at infinitely large values of $x$, whereas the symbol $t$ in the expression $\Sigma a_i t^i$ represents an infinitesimal. Moreover, the homomorphism $\Phi$ introduced in Section 3.6 identifies the field element $\Sigma_N a_i [\rho]^i$ of the nonarchimedean field ${}^\rho\mathcal{R}$ with $\Sigma_N a_i t^i \in L$. So, via $\Phi$, we can regard the asymptotic series $\Sigma a_i x^{-i}$ as an element of ${}^\rho R$. In this sense, each asymptotic power series may be regarded as a field element of both nonarchimedean fields $\mathcal{L}$ and ${}^\rho\mathcal{R}$.

## 3. Nonstandard criterion for asymptotic expansions

Here we shall restrict ourselves to asymptotic sequences of the form $(x^{-\nu_i})$, where the standard sequence $(\nu_i)$ is strictly increasing and $\lim(\nu_i) = \infty$; i.e.:

(1) $\nu_n < \nu_{n+1}$ for each $n \in N$;

(2) $\forall B\, \exists q\, \forall m\, [m > q \to \nu_m > B]$, $B \in R$ and $q, m \in N$.

These conditions on $(\nu_i)$ ensure that the corresponding sequence of standard functions $(x^{-\nu_i})$ is asymptotic (see Section 1).

Our purpose is to develop a criterion in terms of concepts of nonstandard analysis, which will allow us to decide whether an asymptotic sequence $(x^{-\nu_i})$, where $(\nu_i)$ meets the above conditions, yields an asymptotic expansion for a specified function $f$.

The following criterion for the limit of a function at $\infty$ is analogous to the nonstandard criterion for the limit of a sequence (see Section 2.8).

3.1. LEMMA. *Let* $f \in F$ *and* $L \in R$; *then* $\lim_\infty f = L$ *iff* $f(\kappa) \simeq L$ *for each infinite and positive* $\kappa$.

*Proof.* (i) Assume $\lim_\infty f = L$. Choose $h > 0$, $h \in R$. By assumption, there is a positive standard number $q$ such that

$$\forall t\,[t > q \rightarrow |f(t) - L| < h], \quad t \in R,$$

is true for $\mathcal{R}$; therefore this statement is true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$. But $q \in R$; so each positive infinite number is greater than $q$. Thus

$$\forall \kappa\,[\kappa \text{ infinite and positive} \rightarrow |f(\kappa) - L| < h]$$

is true for $^*\mathcal{R}$. Here $h$ is any positive standard number; so $f(\kappa) \simeq L$ whenever $\kappa$ is infinite and positive.

(ii) Assume that $f(\kappa) \simeq L$ for each positive infinite $\kappa$. Then

$$\forall \kappa\,[\kappa > \omega \rightarrow f(\kappa) \simeq L] \quad \kappa \in {}^*R$$

is true for $^*\mathcal{R}$, since $\omega$ is positive and infinite. So

$$\exists q \forall t\,[t > q \rightarrow f(t) \simeq L] \quad q, t \in {}^*R$$

is true for $^*\mathcal{R}$. Thus, for each positive standard $h$,

$$\exists q \forall t\,[t > q \rightarrow |f(t) - L| < h] \quad q, t \in {}^*R$$

is true for $^*\mathcal{R}$, so is true for $\mathcal{R}$ when interpreted in $\mathcal{R}$. Here $h$ is any positive standard number; so

$$\forall h \exists q \forall t\,[t > q \rightarrow |f(t) - L| < h] \quad h > 0,\ q, t \in R$$

is true for $\mathcal{R}$; i.e., $\lim_\infty f = L$. This completes our proof. □

In a moment we shall use Lemma 3.1 to obtain our Nonstandard Criterion for Asymptotic Expansions. First we express our criterion in the language of $\mathcal{R}$.

3.2. CRITERION FOR ASYMPTOTIC EXPANSIONS. $f \sim \Sigma a_i x^{-\nu_i}$ *iff corresponding to each standard number t there is a standard natural number q such that for each* $n > q$, $n \in N$,

$$\lim_\infty x^t \left( f - \sum_0^n a_i x^{-\nu_i} \right) = 0. \tag{3.3}$$

*Proof.* (i) Assume that $f \sim \Sigma a_i x^{-\nu_i}$. Let $t \in R$, and choose $q$ so that $\nu_q > t$. By assumption,

$$\lim_\infty x^{\nu_q} \left( f - \sum_0^q a_i x^{-\nu_i} \right) = 0.$$

But $t < \nu_q$; so

$$\lim_{\infty} x^t \left(f - \sum_0^q a_i x^{-\nu_i}\right) = 0. \tag{3.4}$$

Thus, for each $n > q$, $n \in N$,

$$\lim_{\infty} x^t \left(f - \sum_0^n a_i x^{-\nu_i}\right) = \lim_{\infty} x^t \left(- \sum_{q+1}^n a_i x^{-\nu_i}\right) \qquad \text{by (3.4)}$$

$$= - \lim_{\infty} \sum_{q+1}^n a_i x^{t-\nu_i} = 0\,,$$

since $t < \nu_{q+1}, \ldots, t < \nu_n$. This establishes (3.3).

(ii) Assume that the criterion (3.3) is satisfied. Let $n \in N$; we must show that

$$\lim_{\infty} x^{\nu_n} \left(f - \sum_0^n a_i x^{-\nu_i}\right) = 0.$$

From (3.3), with $t = \nu_n$, we obtain

$$\exists q \forall m \left[m > q \rightarrow \lim_{\infty} x^{\nu_n} \left(f - \sum_0^m a_i x^{-\nu_i}\right) = 0\right] \quad (q, m \in N). \tag{3.5}$$

Take $m$ greater than both $q$ and $n$; then

$$\lim_{\infty} x^{\nu_n} \left(f - \sum_0^n a_i x^{-\nu_i}\right)$$

$$= \lim_{\infty} x^{\nu_n} \left(f - \sum_0^m a_i x^{-\nu_i} + a_{n+1} x^{-\nu_{n+1}} + \ldots + a_m x^{-\nu_m}\right)$$

$$= \lim_{\infty} x^{\nu_n} \left(f - \sum_0^m a_i x^{-\nu_i}\right) + \lim_{\infty} x^{\nu_n} (a_{n+1} x^{-\nu_{n+1}} + \ldots + a_m x^{-\nu_m})$$

$$= \lim_{\infty} x^{\nu_n} (a_{n+1} x^{-\nu_{n+1}} + \ldots + a_m x^{-\nu_m}) \qquad \text{by (3.5)}$$

$$= 0$$

since $\nu_i > \nu_n$ if $i > n$. This proves that $f \sim \Sigma a_i x^{-\nu_i}$. □

We now express this criterion in terms of Nonstandard Analysis.

3.6. NONSTANDARD CRITERION FOR ASYMPTOTIC EXPANSIONS. *$f \sim \Sigma a_i x^{-\nu_i}$ iff corresponding to each standard number $t$ there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and for each infinite positive number $\kappa$,*

$$\kappa^t \left( f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i} \right) \simeq 0. \tag{3.7}$$

*Proof.* Apply Lemma 3.1 to the preceding criterion. □

In symbols, our criterion for $f \sim \Sigma a_i x^{-\nu_i}$ is

$$\forall t\, \exists q\, \forall n \kappa\, [n > q \rightarrow \kappa^t\, [f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i}] \simeq 0] \quad (t \in R,\ q, n \in N, \tag{3.8}$$

$\kappa$ positive and infinite).

Loosely put, if we take enough terms of $\Sigma a_i \kappa^{-\nu_i}$, where $\kappa$ is positive and infinite, the resulting partial sum $\Sigma_0^n a_i \kappa^{-\nu_i}$ is so close to $f(\kappa)$ that multiplying the difference by the infinite number $\kappa^t$ yields an infinitesimal.

One advantage of our criterion is that it is suitable for *all* asymptotic sequences $(x^{-\nu_i})$, of the sort considered, simultaneously.

We can formulate our criterion in a slightly different manner, as follows.

3.9. ALTERNATIVE FORMULATION OF CRITERION. *$f \sim \Sigma a_i x^{-\nu_i}$ iff corresponding to each standard number $t$ there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and for each positive infinitesimal $\epsilon$,*

$$\left| f(1/\epsilon) - \sum_0^n a_i \epsilon^{\nu_i} \right| < \epsilon^t. \tag{3.10}$$

*Proof.* (i) Let $f \sim \Sigma a_i x^{-\nu_i}$. By our criterion, corresponding to each $t \in R$ there is a $q \in N$ such that for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,

$$\kappa^t \left( f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i} \right) \simeq 0,$$

so

$$\kappa^t \left| f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i} \right| < 1,$$

thus

$$(3.11) \qquad \left| f(1/\epsilon) - \sum_0^n a_i \epsilon^{\nu_i} \right| < \epsilon^t,$$

where $\epsilon = 1/\kappa$ is a positive infinitesimal. Since there is a one–one correspondence between positive infinite numbers and positive infinitesimals, this establishes the necessity of our alternative formulation.

(ii) Suppose that the criterion (3.10) is satisfied. Using $t + 1$ in place of $t$, there is a standard natural number $q$ such that for each $n > q$ and for each positive infinitesimal $\epsilon$,

$$\left| f(1/\epsilon) - \sum_0^n a_i \epsilon^{\nu_i} \right| < \epsilon^{t+1},$$

so

$$\kappa^t \left| f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i} \right| < \epsilon,$$

where $\kappa = 1/\epsilon$. Thus

$$\kappa^t \left| f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i} \right| \simeq 0.$$

We conclude that $f \sim \Sigma\, a_i x^{-\nu_i}$ by our Nonstandard Criterion for Asymptotic Expansions. □

We now introduce a family of maps of $^*R - \{0\}$ into $^*R$, one for each positive infinite number $\kappa$. Thus let $V_\kappa$ be the map of $^*R - \{0\}$ into $^*R$ for which $V_\kappa(a) = \log_\epsilon |a|$ whenever $a \neq 0$, where $\epsilon = 1/\kappa$. Now for each $a \neq 0$,

$$\log_\epsilon |a| = \frac{\ln |a|}{\ln \epsilon};$$

thus $\log_\epsilon$ is monotonically decreasing, since $\ln \epsilon$ is negative and the function ln is monotonically increasing.

Therefore, if

$$(3.12) \qquad \left| f(\kappa) - \sum_0^n a_i \epsilon^{\nu_i} \right| < \epsilon^t$$

and the LHS of (3.12) is not zero, then

$$\log_\epsilon \left| f(\kappa) - \sum_0^n a_i \epsilon^{\nu_i} \right| > \log_\epsilon \epsilon^t;$$

i.e.,

$$V_\kappa \left( f(\kappa) - \sum_0^n a_i \epsilon^{\nu_i} \right) > t. \tag{3.13}$$

Applying our Criterion 3.9, this yields:

3.14. LEMMA. *$f \sim \Sigma a_i x^{-\nu_i}$, provided that corresponding to each standard number $t$ there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,*

$$V_\kappa \left( f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i} \right) > t.$$

To obtain the converse of this lemma we want each map $V_\kappa$ to associate $\infty$ with 0, where we regard $\infty$ as greater than each standard number; this makes (3.13) true if $f(\kappa) = \Sigma_0^n a_i \kappa^{-\nu_i}$. Extending our maps $V_\kappa$ in this way yields:

3.15. CRITERION. *$f \sim \Sigma a_i x^{-\nu_i}$ iff corresponding to each standard number $t$ there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,*

$$V_\kappa \left( f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i} \right) > t.$$

Notice that if $V_\kappa(f(\kappa))$ is positive and infinite, then $\log_\epsilon |f(\kappa)| > m$ for each $m \in N$; i.e., $|f(\kappa)| < \epsilon^m$ for each $m \in N$. Therefore

$$\left| f(\kappa) - \sum_0^n 0\kappa^{-\nu_i} \right| > \epsilon^m$$

for each $n,m \in N$, where $(x^{-\nu_i})$ is any asymptotic sequence. Applying Criterion 3.9, we conclude that $f \sim \Sigma\, 0x^{-\nu_i}$. Conversely, if $f \sim \Sigma\, 0x^{-\nu_i}$, then by Criterion 3.15, $V_\kappa(f(\kappa)) > t$ whenever $t \in R$ and $\kappa$ is positive and infinite; i.e., $V_\kappa(f(\kappa))$ is positive and infinite in this case. This establishes our next lemma.

3.16. LEMMA. $f \sim \Sigma\, 0x^{-\nu_i}$ *iff* $V_\kappa(f(\kappa))$ *is positive and infinite whenever* $\kappa$ *is positive and infinite.*

Denoting any asymptotic expansion of the form $\Sigma\, 0x^{-\nu_i}$ by "0", our result can be put as follows: $f \sim 0$ iff $f(\kappa) \approx 0$ with respect to $1/\kappa$, whenever $\kappa$ is positive and infinite.

It is natural to regard functions $f$ and $g$ as equivalent, i.e., *equal* from the viewpoint of asymptotic expansions, if they possess a common asymptotic expansion; in this case, we write $f \cong g$. For example, $(1+x)^{-1}$ and $(1+x)^{-1} + e^{-x}$ have a common asymptotic expansion, namely $\Sigma(-1)^i x^{-i-1}$, so $(1+x)^{-1} \cong (1+x)^{-1} + e^{-x}$. Clearly $\cong$ is an equivalence relation on the set of functions that possess asymptotic expansions.

Summarizing, $f \cong g$ iff

(a) $f - g \sim 0$ (here 0 denotes $\Sigma\, 0x^{-\nu_i}$);

(b) both $f$ and $g$ possess an asymptotic expansion.

There is, however, a great deal to be gained by dropping the second requirement, i.e., that both $f$ and $g$ possess asymptotic expansions. Accordingly, we now extend the notion of *equivalent* functions as follows.

3.17. DEFINITION. $f \cong g$ if $f - g \sim 0$.

In view of Lemma 3.16 we can characterize our extended equivalence relation as follows.

3.18. CRITERION FOR EQUIVALENT FUNCTIONS. $f \cong g$ *iff* $f(\kappa) - g(\kappa)$ *is an iota with respect to* $1/\kappa$ *for each positive infinite* $\kappa$.

We have just seen that $f(\kappa)$ is an iota, with respect to $1/\kappa$, if $V_\kappa(f(\kappa))$ is positive and infinite. Similarly, if $V_\kappa(f(\kappa))$ is negative and infinite, it is easy to see that $f(\kappa)$ is a mega with respect to $\kappa$. Finally, if $V_\kappa(f(\kappa))$ is finite, then $f(\kappa) \in M_0$ (where $\rho = 1/\kappa$); so there is a standard natural number $n$ such that $\rho^{n+1} < |f(\kappa)| < \rho^n$.

We want to extend each map $V_\kappa$ to a valuation $v_\kappa$ on the field ${}^\rho\mathfrak{R}$, where $\rho = 1/\kappa$. Since ${}^0(\log_\rho |a|) = {}^0(\log_\rho |b|)$ if $a - b \in M_1$, define for each $[a] \in {}^\rho R$, $[a] \neq M_1$,

$$v_\kappa([a]) = {}^0V_\kappa(a) = {}^0(\log_\rho |a|),$$

and define $v_\kappa([0]) = \infty$. We point out that each $v_\kappa$ is a nonarchimedean valuation on ${}^\rho\mathfrak{R}$ $(\rho = 1/\kappa)$.

It is convenient to write $v_\kappa(a)$ in place of $v_\kappa([a])$; i.e., we wish to apply $v_\kappa$

to a member of a coset, say $a$, rather than the coset $[a]$. So we define for each $a \in {}^*R$, $v_\kappa(a) = v_\kappa([a])$; i.e.,

$$v_\kappa(a) = \begin{cases} {}^0(\log_\rho |a|) & \text{if } a \notin [0], \\ \infty & \text{if } a \in [0]. \end{cases}$$

In Chapter 7 we shall use these valuations in out study of an important function space. In that connection, we present the following fact about $|\ |_{v_\kappa}$, where $\kappa$ is any positive infinite number, the associated map of ${}^\rho R$ into $R$.

3.19. LEMMA. *Let $a \in {}^*R$, $a \neq 0$, and let $\kappa$ be positive and infinite. Then*

$$|a|_{v_\kappa} = {}^0(|a|^{1/(\ln \kappa)}).$$

*Proof.* (i) Assume that $a \in [0]$, i.e., $a \in M_1$ (where $\rho = 1/\kappa$). Then for each $n \in N$, $|a| < \rho^n$, so $\ln |a| < n \ln \rho$; thus

$$\frac{\ln |a|}{\ln \kappa} < \frac{n \ln \rho}{\ln \kappa} = -n$$

since $\ln \rho = -\ln \kappa$. So, for each $n \in N$,

$$|a|^{1/(\ln \kappa)} = \exp\left(\frac{\ln |a|}{\ln \kappa}\right) < e^{-n}$$

This proves that $|a|^{1/(\ln \kappa)}$ is an infinitesimal. Therefore

$${}^0(|a|^{1/(\ln \kappa)}) = 0.$$

But $|a|_{v_\kappa} = 0$ since $v_\kappa(a) = \infty$.

(ii) Assume that $a \notin [0]$. Then

$$v_\kappa(a) = {}^0(\log_\rho |a|) = {}^0\left(\frac{\ln |a|}{\ln \rho}\right) = \frac{\ln |a|}{\ln \rho} + \eta,$$

where $\eta \simeq 0$. So

$$-v_\kappa(a) = \frac{\ln |a|}{\ln \kappa} - \eta;$$

thus

$$|a|_{v_\kappa} = e^{-v_\kappa(a)} = \exp\left[\frac{\ln |a|}{\ln \kappa} - \eta\right] = e^{-\eta}\, |a|^{1/(\ln \kappa)}.$$

We can simplify the expression on the right by applying an elementary fact in the arithmetic of infinitesimals. Let $s$ and $t$ be numbers such that $s \simeq 1$ and

$st \in R$; then $st = {}^0t$. Here $e^{-\eta} \simeq 1$ and $e^{-\eta}\,|a|^{1/(\ln\kappa)} \in R$. Thus

$$e^{-\eta}\,|a|^{1/(\ln\kappa)} = {}^0(\,|a|^{1/(\ln\kappa)}).$$

We conclude that $|a|_{v_\kappa} = {}^0(\,|a|^{1/(\ln\kappa)})$. □

## 4. Watson's Lemma

To illustrate our nonstandard methods we now prove a well-known result concerning asymptotic representations. First we shall need the following lemmas.

4.1. LEMMA. $x^a < e^{bx}$ *if a and b are finite and positive.*

4.2. COMMENT. This is a statement about functions in a Hardy space and means that $t^a < e^{bt}$ if $t$ is sufficiently large, $t \in R$.

We shall use Lemma 4.1 to establish our next lemma.

4.3. LEMMA. *Let $t,m,Z$ be finite, and let $\kappa$ be positive and infinite. Then*

$$\left|\int_Z^\infty e^{-\kappa z}\, z^m\right| < \kappa^{-t}.$$

*Proof.* $|z|^m < e^{\kappa z/2}$ by Lemma 4.1; so

$$\left|\int_Z^\infty e^{-\kappa z}\, z^m\right| < \int_Z^\infty e^{-\kappa z/2} = \frac{2}{\kappa}\, e^{-\kappa Z/2}.$$

Of course, our lemma is obvious if $t \leqslant 0$; accordingly, we can assume that $t > 0$. In this case, by Lemma 4.1, $e^{-Zx/2} < x^{-t}$; so $e^{-Z\kappa/2} < \kappa^{-t}$ for each positive infinite $\kappa$. Thus

$$\left|\int_Z^\infty e^{-\kappa z}\, z^m\right| < \frac{2}{\kappa}\,\kappa^{-t} < \kappa^{-t}.\ \square$$

We can extend this result a little.

4.4. COROLLARY. *Let a be finite. Then*

$$\left| \int_Z^\infty a\, \mathrm{e}^{-\kappa z} z^m \right| < \kappa^{-t}$$

*for each finite t.*

The main concern of this section is Watson's Lemma, which we now present.

4.5. WATSON'S LEMMA. *Let f be a complex function which is analytic in a neighbourhood of the origin, let $m > -1$, $m \in R$, and let $Z \in R$ be any positive finite number within the circle of convergence of $\Sigma_N a_i z^i$, a power series expansion of f about the origin. Let g be the function such that for $t \in R$, t large and positive, $g(t) = \int_0^Z \mathrm{e}^{-tz} z^m f$, where the path of integration is along the positive real axis. Then*

$$g \sim \Sigma\, a_i \Gamma(m+i+1) x^{-m-i-1}.$$

*Proof.* First, we point out that the given asymptotic expansion of $g$ is obtained from the definition of $g$ by replacing $f$ by $\Sigma_N a_i z^i$, replacing $Z$ by $\infty$, and integrating term by term. Carrying out this prescription, for each $i \in N$ let $g_i$ be the function such that

$$g_i(t) = \int_0^\infty \mathrm{e}^{-tz} z^{m+i} = \Gamma(m+i+1)/t^{m+i+1},$$

so

$$\Sigma\, a_i g_i = \Sigma\, a_i \Gamma(m+i+1) x^{-m-i-1},$$

which has the required form $\Sigma\, b_i x^{-\nu_i}$. To prove that $g \sim \Sigma\, a_i g_i$ we shall apply our Nonstandard Criterion for Asymptotic Expansions 3.6. Accordingly, let $\kappa$ be positive and infinite, and consider:

$$\begin{aligned} g(\kappa) - \sum_0^n a_i g_i(\kappa) &= \int_0^Z \mathrm{e}^{-\kappa z} z^m f - \int_0^\infty \mathrm{e}^{-\kappa z} z^m \sum_0^n a_i z^i \\ &= \int_0^\infty \mathrm{e}^{-\kappa z} z^m \left[ f - \sum_0^n a_i z^i \right] - \int_Z^\infty \mathrm{e}^{-\kappa z} z^m \left[ f - \sum_0^n a_i z^i \right] \\ &\quad - \int_Z^\infty \mathrm{e}^{-\kappa z} z^m \sum_0^n a_i z^i. \end{aligned}$$

For each $i \in N$, let

$$\mu_i = \int_Z^\infty e^{-\kappa z} z^{m+i}.$$

Then

$$(4.6) \qquad g(\kappa) - \sum_0^n a_i g_i(\kappa) = \int_0^\infty e^{-\kappa z} z^m \left[ f - \sum_0^n a_i z^i \right]$$

$$- \int_Z^\infty e^{-\kappa z} z^m \left[ f - \sum_0^n a_i z^i \right] - \sum_0^n a_i \mu_i.$$

Recall that $f = \Sigma_N a_i z^i$; so, if $n$ is sufficiently large,

$$(4.7) \qquad \left| z^m \left( f - \sum_0^n a^i z^i \right) \right| = |z^{m+n+1}| \left| \sum_{n+1}^\infty a^i z^{i-n-1} \right| \leqslant B |z^{m+n+1}|$$

where $B \in R$. From (4.6),

$$\left| g(\kappa) - \sum_0^n a_i g_i(\kappa) \right| \leqslant B \int_0^\infty e^{-\kappa z} z^{m+n+1} + B \int_Z^\infty e^{-\kappa z} z^{m+n+1} + \sum_0^n |a_i \mu_i|$$

$$< B\,\Gamma(m+n+2)/\kappa^{m+n+2} + \kappa^{-t-3} + (n+1)\kappa^{-t-3}$$

by Corollary 4.4

$$< B\,\Gamma(m+n+2)/\kappa^{m+n+2} + \kappa^{-t-2}$$

for each finite $t$. Clearly, given $t \in R$ we can choose $n$ so that both $m + n + 2 > t + 2$ (i.e., $n > t - m$) and $n$ is large enough to satisfy (4.7). For any such $n$,

$$B\,\Gamma(m+n+2)/\kappa^{m+n+2} + \kappa^{-t-2} < B\,\Gamma(m+n+2)\,\kappa^{-t-2} + \kappa^{-t-2} < \kappa^{-t-1},$$

so

$$\left| g(\kappa) - \sum_0^n a_i g_i(\kappa) \right| < \kappa^{-t-1},$$

thus

$$\kappa^t \left( g(\kappa) - \sum_0^n a_i g_i(\kappa) \right) \simeq 0.$$

We conclude that $g \sim \Sigma a_i g_i$. This completes our proof. □

Taking $m = 0$ in Watson's Lemma yields the following. Let $g$ be the function such that for each $t$,

$$g(t) = \int_0^Z e^{-tz} f,$$

where $f = \Sigma a_i z^i$ throughout a neighbourhood of the origin and $Z$ is a finite number within its circle of convergence. For each $i \in N$, let $g_i$ be the function such that for each $t$,

$$g_i(t) = \int_0^\infty e^{-tz} z^i = \Gamma(i+1)/t^{i+1} = i!\, t^{-i-1},$$

i.e., $g_i = i!\, x^{-i-1}$. Thus, by Watson's Lemma,

$$g \sim \Sigma\, i!\, a_i x^{-i-1} = \frac{a_0}{x} + \frac{a_1}{x^2} + \ldots + i!\frac{a_i}{x^{i+1}} + \ldots .$$

Moreover, let $f$ be an entire function that possesses a Laplace transform $L[f] = \lim_{Z\to\infty} g$. We claim that $\Sigma\, i!\, a_i x^{-i-1}$ is also an asymptotic expansion of $L[f]$, provided that $f$ is bounded on the positive real axis by some $x^s$, i.e., $f = O(x^s)$, where $s \in R$. In this case, for $Z$ sufficiently large, the remainder term

$$\left|\int_Z^\infty e^{-\kappa z} f\right| \leqslant \left|\int_Z^\infty e^{-\kappa z} z^s\right| < \kappa^{-t},$$

by Lemma 4.3, for each $t \in R$, where $\kappa$ is positive infinite. Thus, for each $t \in R$,

$$\left|(L[f])(\kappa) - \sum_0^n i!\, a_i \kappa^{-i}\right| \leqslant |(L[f])(\kappa) - g(\kappa)| + \left|g(\kappa) - \sum_0^n i!\, a_i \kappa^{-i}\right|$$
$$< \kappa^{-t-2} + \kappa^{-t-2} < \kappa^{-t-1}.$$

So, for each $t \in R$ and each positive infinite $\kappa$,

$$\kappa^t\left[(L[f])(\kappa) - \sum_0^n i!\, a_i \kappa^{-i-1}\right] \simeq 0,$$

provided that $n$ is sufficiently large. We conclude that $L[f] \sim \Sigma\, i!\, a_i x^{-i-1}$.

## 5. Other scales

In place of the scale of comparison provided by the asymptotic sequence involved, which we have used so far, it is also possible under certain circumstances to characterize asymptotic expansions in terms of the scale provided by a second asymptotic sequence.

Let $\phi = (\phi_i)$ and $\psi = (\psi_i)$ be asymptotic sequences. We shall say that $\Sigma a_i \phi_i$ is asymptotic to $f$ with respect to $\psi$, and write $f \sim_\psi \Sigma a_i \phi_i$, provided that for each $t \in N$,

$$f - \sum_0^n a_i \phi_i = o(\psi_t),$$

if $n$ is sufficiently large; in symbols,

$$\forall t\, \exists q\, \forall n\, [n > q \rightarrow f - \sum_0^n a_i \phi_i = o(\psi_t)].$$

For example, let $\psi = (x^{-2\nu_i})$, where $(\nu_i)$ is strictly increasing and unbounded, and let $f \sim \Sigma a_i x^{-\nu_i}$. Then it is easy to see that $f \sim_\psi \Sigma a_i x^{-\nu_i}$.

Here is another example of this idea.

5.1. EXAMPLE. Let $\psi = (x^i \mathrm{e}^{-ix})$, and let $f \sim \Sigma a_i \mathrm{e}^{-ix}$. By assumption, and in view of Lemma 3.1,

$$\mathrm{e}^{\kappa n}\left[f(\kappa) - \sum_0^n a_i \mathrm{e}^{-\kappa i}\right] \simeq 0$$

for any $n \in N$ and for any positive infinite $\kappa$. Thus, given $t \in N$,

$$\mathrm{e}^{\kappa t}\left[f(\kappa) - \sum_0^n a_i \mathrm{e}^{-\kappa i}\right] \simeq 0$$

for any $n > t$, $n \in N$, and for any positive infinite $\kappa$. So

$$\kappa^{-t}\, \mathrm{e}^{\kappa t}\left[f(\kappa) - \sum_0^n a_i \mathrm{e}^{-\kappa i}\right] \simeq 0,$$

where $n > t$ and $\kappa$ is positive and infinite; by Lemma 3.1,

$$\lim_{\infty} \frac{f - \Sigma_0^n a_i \mathrm{e}^{-ix}}{x^t \mathrm{e}^{-tx}} = 0.$$

We conclude that $f \sim_{\psi} \Sigma a_i \mathrm{e}^{-ix}$.

To bring out some basic techniques, and at the same time to show that our notion is a true generalization of the concept of an asymptotic expansion, we now prove the following fact.

5.2. LEMMA. *$f \sim_{\phi} \Sigma a_i \phi_i$ iff $f \sim \Sigma a_i \phi_i$.*

*Proof.* Assume that $f \sim_{\phi} \Sigma a_i \phi_i$. Then for each $m \in N$ there is a $q \in N$ such that $f - \Sigma_0^n a_i \phi_i = \mathrm{o}(\phi_m)$ if $n > q$. We can assume that $n > m$; thus

$$f - \sum_0^m a_i \phi_i - \sum_{m+1}^n a_i \phi_i = \mathrm{o}(\phi_m).$$

But $\Sigma_{m+1}^n a_i \phi_i = \mathrm{o}(\phi_m)$ since $\phi$ is an asymptotic sequence. Thus $f - \Sigma_0^m a_i \phi_i = \mathrm{o}(\phi_m)$ for each $m \in N$; so $f \sim \Sigma a_i \phi_i$.

Next assume that $f \sim \Sigma a_i \phi_i$. Then, for each $m \in N$, $f - \Sigma_0^m a_i \phi_i = \mathrm{o}(\phi_m)$. Take $n > m$; then $\phi_{m+j} = \mathrm{o}(\phi_m)$ for $j = 1, 2, 3, \ldots$. So

$$\sum_{m+1}^n a_i \phi_i = \mathrm{o}(\phi_m),$$

and it follows that

$$f - \sum_0^m a_i \phi_i - \sum_{m+1}^n a_i \phi_i = \mathrm{o}(\phi_m),$$

i.e., $f - \Sigma_0^n a_i \phi_i = \mathrm{o}(\phi_m)$ for each $n > m$. We conclude that $f \sim_{\phi} \Sigma a_i \phi_i$. This completes our proof. □

We say that $f < g$ provided that there is a neighbourhood of $\infty$, i.e., a semi-infinite interval (see Section 1.6), say $N_{\infty}$, such that $f(t) < g(t)$ for each $t \in N_{\infty}$. We define the relation $>$ similarly; i.e., $f > g$ provided there is a neighbourhood of $\infty$, say $N_{\infty}$, such that $f(t) > g(t)$ for each $t \in N_{\infty}$. We shall use $\leqslant$ in the obvious way; i.e., $f \leqslant g$ iff $f(t) \leqslant g(t)$ for each $t \in N_{\infty}$, some neighbourhood of $\infty$.

The following idea is useful. We shall say that sequences of functions $(\phi_i)$ and $(\psi_i)$ are *comparable* provided that the absolute value of each term of either sequence is greater than the absolute value of some term of the other sequence; in symbols,

$$\forall i\, \exists j\, [|\phi_i| > |\psi_j|], \qquad \forall j\, \exists i\, [|\psi_j| > |\phi_i|] \qquad (i,j \in N).$$

For example, the sequences $(e^{-ix})$ and $(x^i e^{-ix})$ are comparable, as are the sequences $(x^{-\nu_i})$ and $(x^{-2\nu_i})$, where $(\nu_i)$ is strictly increasing and unbounded.

Our main point is to prove that for the case of comparable sequences, an asymptotic expansion involving one of the sequences, can be taken with respect to either scale.

5.3. THEOREM. *Let $\phi = (\phi_i)$ and $\psi = (\psi_i)$ be comparable asymptotic sequences. Then $f \sim_\phi \Sigma\, a_i \phi_i$ iff $f \sim_\psi \Sigma\, a_i \phi_i$.*

*Proof.* In view of Lemma 5.2, it is enough to show that $f \sim \Sigma\, a_i \phi_i$ iff $f \sim_\psi \Sigma\, a_i \phi_i$.

(i) Assume that $f \sim \Sigma\, a_i \phi_i$. For each $t \in N$ we can choose $q$ so that $\psi_t > \phi_q$. Take $n > q$ and consider $f - \Sigma_0^n\, a_i \phi_i$; now

$$\left|\frac{f - \Sigma_0^n a_i \phi_i}{\psi_t}\right| \leqslant \left|\frac{f - \Sigma_0^n a_i \phi_i}{\phi_q}\right|$$

$$\leqslant \left|\frac{f - \Sigma_0^q a_i \phi_i}{\phi_q}\right| + \left|\frac{\Sigma_{q+1}^n a_i \phi_i}{\phi_q}\right|.$$

By assumption, $f - \Sigma_0^q\, a_i \phi_i = o(\phi_q)$; moreover, $\phi_{q+j} = o(\phi_q)$ for $j = 1,2,3,\ldots$ since $\phi$ is an asymptotic sequence. Thus

$$\lim \left(\frac{f - \Sigma_0^n a_i \phi_i}{\psi_t}\right) = 0;$$

so $f - \Sigma_0^n\, a_i \phi_i = o(\psi_t)$ if $n > q$. This proves that $f \sim_\psi \Sigma\, a_i \phi_i$.

(ii) Assume that $f \sim_\psi \Sigma\, a_i \phi_i$. We shall show that $f - \Sigma_0^n\, a_i \phi_i = o(\phi_n)$ for each $n \in N$. Choose $n \in N$; by assumption, $\phi_n > \psi_t$ for some $t \in N$. Now, for $m$ sufficiently large, $f - \Sigma_0^m\, a_i \phi_i = o(\psi_t)$; so $f - \Sigma_0^m\, a_i \phi_i = o(\phi_n)$. We can assume that $m > n$, so

$$\lim \left(\frac{f - \Sigma_0^n a_i \phi_i}{\phi_n} + \frac{\Sigma_{n+1}^m a_i \phi_i}{\phi_n}\right) = 0$$

and it follows that $f - \Sigma_0^n a_i \phi_i = o(\phi_n)$. We conclude that $f \sim \Sigma a_i \phi_i$. This completes our proof. □

## 6. A generalized criterion for asymptotic expansions

In this section we shall formulate two conditions on an asymptotic sequence $(\phi_i)$ which together ensure that $f \sim \Sigma a_i \phi_i$ iff the following condition is met:

(6.1) For each $t \in R$ there is a $q \in N$ such that for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,

$$\kappa^t \left[ f(\kappa) - \sum_0^n a_i \phi_i(\kappa) \right] \simeq 0.$$

That is, if $(\phi_i)$ satisfies both conditions, then $f \sim \Sigma a_i \phi_i$ iff

$$\forall t \, \exists q \, \forall n \kappa \left[ n > q \to \kappa^t \left[ f(\kappa) - \sum_0^n a_i \phi_i(\kappa) \right] \simeq 0 \right],$$

where the domain of the quantifiers is given by (6.1). We have already carried out a program of this sort; indeed, in Section 6.3 we showed that (6.1) is a criterion for $f \sim \Sigma a_i \phi_i$ provided that the asymptotic sequence $(\phi_i)$ has the form $(x^{-\nu_i})$, where $(\nu_i)$ is strictly increasing and $\lim(\nu_i) = \infty$.

Our goal, here, is to extend, indeed to characterize, the class of asymptotic sequences such that (6.1) provides a criterion for asymptotic expansions.

Throughout this section, $(\phi_i)$ is an asymptotic sequence. As usual, we shall write $f < g$, where $f$ and $g$ are functions, provided that for some $a \in R$,

$$\forall t \, [t > a \wedge t \in R \to f(t) < g(t)].$$

We now exhibit a condition on $(\phi_i)$ so that $f \sim \Sigma a_i \phi_i$ if both $f$ and $\Sigma a_i \phi_i$ satisfy (6.1).

6.2. LEMMA. *$f \sim \Sigma a_i \phi_i$ if*

(a) $\forall n \, \exists t \, [x^{-t} < |\phi_n|]$, *$n \in N$ and $t \in R$;*

(b) $\forall t \, \exists q \, \forall n \kappa \, [n > q \to \kappa^t \, [f(\kappa) - \Sigma_0^n a_i \phi_i(\kappa)] \simeq 0]$, *$t \in R$, $q, n \in N$, and $\kappa$ positive and infinite.*

*Proof*. Choose $n \in N$; by (a) there is a standard number $t$ such that $x^{-t} < |\phi_n|$. By (b), with $t+1$ for its first placeholder, there is a standard natural number $q$, $q > n$, such that

$$\forall m\kappa \left[ m > q \to \kappa^{t+1} \left[ f(\kappa) - \sum_0^m a_i \phi_i(\kappa) \right] \simeq 0 \right],$$

where the first quantifier refers to $N$, and $\kappa$ is positive and infinite. Let $m > q$, $m \in N$, and let $\kappa$ be any positive infinite number. Then

$$\left| f(\kappa) - \sum_0^m a_i \phi_i(\kappa) \right| < \kappa^{-t-1} < \frac{|\phi_n(\kappa)|}{\kappa},$$

so

$$\frac{f(\kappa) - \Sigma_0^m a_i \phi_i(\kappa)}{\phi_n(\kappa)} \simeq 0$$

Therefore

$$\lim_\infty \frac{f - \Sigma_0^m a_i \phi_i}{\phi_n} = 0.$$

Since $(\phi_i)$ is asymptotic, $\lim_\infty \phi_{n+j}/\phi_n = 0$ for $j = 1,2,3,\dots$. Thus

$$\begin{aligned} \lim_\infty \frac{f - \Sigma_0^n a_i \phi_i}{\phi_n} &= \lim_\infty \frac{f - \Sigma_0^m a_i \phi_i}{\phi_n} + \lim_\infty \frac{\Sigma_{n+1}^m a_i \phi_i}{\phi_n} \\ &= \lim_\infty \frac{f - \Sigma_0^m a_i \phi_i}{\phi_n} \\ &= 0. \end{aligned}$$

This proves that for each $n \in N$, $f - \Sigma_0^n a_i \phi_i = o(\phi_n)$; we conclude that $f \sim \Sigma a_i \phi_i$. This completes our proof. □

So far we have found one condition on our sequences $(\phi_i)$ – namely, condition (a) of Lemma 6.2 – which ensures that $f \sim \Sigma a_i \phi_i$ in case the function $f$ and the asymptotic series $\Sigma a_i \phi_i$ satisfy our criterion (6.1). We now present a second condition on $(\phi_i)$ – namely, condition (ii) of Lemma 6.3 – which ensures that $f$ and $\Sigma a_i \phi_i$ satisfy the criterion (6.1) in case $f \sim \Sigma a_i \phi_i$.

6.3. LEMMA. *$f$ and $\Sigma a_i \phi_i$ satisfy criterion* (6.1) *if*:

(i) $f \sim \Sigma a_i \phi_i$;

(ii) $\forall t \, \exists n \, [|\phi_n| < x^{-t}]$, $t \in R$ *and* $n \in N$.

*Proof.* Choose $t \in R$; by (ii), there is a standard natural number $q$ such that $|\phi_q| < x^{-t}$. For this $q$, by (i),

$$\lim_{\infty} \frac{f - \Sigma_0^q a_i \phi_i}{\phi_q} = 0.$$

Therefore

$$\lim_{\infty} x^t \left(f - \sum_0^q a_i \phi_i\right) = 0$$

since $|\phi_q| < x^{-t}$. Moreover,

$$\phi_{q+j} = o(\phi_q) \quad \text{for } j = 1, 2, 3, \ldots,$$

thus

$$\phi_{q+j} = o(x^{-t}) \quad \text{for } j = 1, 2, 3, \ldots.$$

So, for $m > q$, $m \in N$,

$$\begin{aligned} \lim_{\infty} x^t \left(f - \sum_0^m a_i \phi_i\right) &= \lim_{\infty} x^t \left(f - \sum_0^q a_i \phi_i\right) - \lim_{\infty} \sum_{q+1}^m a_i x^t \phi_i \\ &= \lim_{\infty} x^t \left(f - \sum_0^q a_i \phi_i\right) \qquad \text{since } \phi_{q+j} = o(x^{-t}) \text{ for } j = 1, 2, 3 \ldots \\ &= 0. \end{aligned}$$

Therefore, for each positive infinite $\kappa$,

$$\kappa^t \left[f(\kappa) - \sum_0^m a_i \phi_i(\kappa)\right] \simeq 0$$

if $m > q$, i.e., $f$ and $\Sigma a_i \phi_i$ satisfy criterion (6.1). This completes our proof of the lemma. □

We summarize Lemmas 6.2 and 6.3 in the following theorem.

6.4. THEOREM. *Let* $(\phi_i)$ *be an asymptotic sequence such that*

$$\forall n\, \exists t\, [x^{-t} < |\phi_n|], \qquad \forall t\, \exists n\, [|\phi_n| < x^{-t}] \qquad (n \in N,\ t \in R).$$

*Then* $f \sim \Sigma\, a_i \phi_i$ *iff*

$$\forall t\, \exists q\, \forall m\kappa \left[ m > q \to \kappa^t \left[ f(\kappa) - \sum_0^m a_i \phi_i(\kappa) \right] \simeq 0 \right],$$

*where* $t \in R$, $q, m \in N$, *and* $\kappa$ *positive and infinite.*

Notice that each sequence of the form $(x^{-\nu_i})$, where $(\nu_i)$ is strictly increasing and $\lim(\nu_i) = \infty$, satisfies the two conditions of our theorem.

Let us see if we can simplify this pair of conditions. Now there is no loss in formulating our Nonstandard Criterion for Asymptotic Expansions 3.6, so that $t \in N$ instead of $t \in R$. This allows us to interpret $t$ as a standard natural number, instead of a standard real number, in both of our conditions. In turn, this means that we are comparing the sequence $(|\phi_i|)$ to the fixed sequence $(x^{-i})$; indeed, we require that each term of either sequence is greater than some term of the other sequence. According to the definition of comparable sequences introduced in Section 5, these sequences are comparable. Using this terminology we can rephrase Theorem 6.4 as follows.

6.5. THEOREM. *Let* $(\phi_i)$ *be an asymptotic sequence comparable to* $(x^{-i})$. *Then* $f \sim \Sigma\, a_i \phi_i$ *iff*

$$\forall t\, \exists q\, \forall m\kappa \left[ m > q \to \kappa^t \left[ f(\kappa) - \sum_0^m a_i \phi_i(\kappa) \right] \simeq 0 \right],$$

*where* $t \in R$, $q, m \in N$, $\kappa$ *positive and infinite.*

We mention that the sum of two asymptotic sequences of the form $(x^{-\nu_i})$ is not necessarily of the same form. However, the sum is comparable to $(x^{-i})$.

CHAPTER 7

# POPKEN SPACE

## 1. Asymptotically finite functions

The purpose of this chapter is to study the function space discussed by J. Popken [1953]; much of the basic work in this area was carried out by van der Corput (see [1954a]) and in due course was refined by Popken.

A standard function $f$ is said to be *asymptotically finite* provided that:

(1) dom $f$ includes a neighbourhood of $\infty$, i.e., an infinite subset of the form $\{t \in R \mid t > a\}$ for some $a \in R$;

(2) $f = O(x^t)$ for some $t \in R$.

Recall that $f = O(x^t)$ iff there is a positive standard number $B$ and a neighbourhood $N_\infty$ of $\infty$ such that $|f(c)| < B c^t$ for each $c \in N_\infty$.

The first condition ensures that $\kappa \in \text{dom}\ {}^*f$ if $\kappa$ is positive and infinite. The second condition ensures that $f(\kappa)$ is *not* a mega with respect to $\kappa$; i.e., $f(\kappa) \in M_0$ where $\rho = 1/\kappa$. Therefore, for each positive, infinite $\kappa$, $V_\kappa(f(\kappa))$ is not both positive and infinite; indeed, there is a standard integer $i$ such that

$$0 \leqslant |f(\kappa)| < \rho^i.$$

In particular, $v_\kappa(f(\kappa))$ is defined (see Section 6.3); in fact,

$$v_\kappa(f(\kappa)) = \begin{cases} {}^0(\log_\rho |f(\kappa)|) & \text{if } f(\kappa) \notin [0], \\ \infty & \text{if } f(\kappa) \in [0]. \end{cases}$$

In this function space, which we denote by $\mathcal{P}$, we say that functions $f$ and $g$ are *asymptotically equal*, and write $f \cong g$, if their difference is asymptotic to 0 (Popken writes $f \sim g$). This is the equivalence relation introduced in Section 6.3. Recall Criterion 6.3.18: $f \cong g$ iff for each positive infinite $\kappa$, $f(\kappa) \approx g(\kappa)$ with respect to $1/\kappa$ (in words, $f(\kappa) - g(\kappa)$ is an iota with respect to $1/\kappa$).

For example, each function $a x^t$, where $a, t \in R$, is asymptotically finite. The sum, difference or product of asymptotically finite functions is also asymptotically finite; indeed, asymptotically finite functions form a ring with respect to addition and multiplication. This ring is commutative and has a unit, the standard function $1 = \{(t, 1) \mid t \in R\}$; its zero, i.e., its additive identity, is the standard function $0 = \{(t, 0) \mid t \in R\}$.

We point out that the function space of this section is *not* identical with the space of functions which possess asymptotic expansions of the form $\Sigma a_i x^{-\nu_i}$. For example, the function ln is asymptotically finite since $\ln = O(x)$; let us show that ln does not possess an asymptotic expansion of the required form. Assume that $\ln \sim \Sigma a_i x^{-\nu_i}$, where $(\nu_i)$ is strictly increasing and $\lim(\nu_i) = \infty$. By Lemma 6.1.1,

$$a_0 = \lim_{\infty} x^{\nu_0} \ln = \begin{cases} \infty & \text{if } \nu_0 \geqslant 0, \\ 0 & \text{if } \nu_0 < 0. \end{cases}$$

It follows that $\Sigma\, 0x^{-\nu_i}$ is the only possible asymptotic expansion of ln; but this is out of the question (e.g., apply Lemma 6.3.16).

Bear in mind that the objects of our function space are equivalence classes of asymptotically finite functions, e.g.,

$$\{f \mid f \cong x^2 \text{ and } f \text{ is asymptotically finite}\}.$$

Our use of the equivalence relation $\cong$, i.e., the notion of *asymptotically equal* functions, is designed to simplify our presentation by mentioning functions rather than equivalence classes.

It is helpful to characterize the big O relation in nonstandard terms.

1.1. THEOREM. *For each $t \in R$, $f = O(x^t)$ iff there is a standard number $B$ such that $\forall \kappa\ [|f(\kappa)| < B\kappa^t]$, $\kappa$ positive and infinite.*

*Proof.* (i) Assume that $f = O(x^t)$, where $t \in R$. Then there are standard numbers $a$ and $B$ such that

$$\forall x\ [x > a \to |f(x)| < Bx^t]$$

is true for $\mathcal{R}$, so is true for $^*\mathcal{R}$ when interpreted in $^*\mathcal{R}$. In particular,

$$\forall \kappa\ [|f(\kappa)| < B\kappa^t]$$

since $\kappa > a$ for each positive infinite $\kappa$.

(ii) Assume that $\forall \kappa\ [|f(\kappa)| < B\kappa^t]$ is true for $^*\mathcal{R}$, where $B$ and $t$ are standard numbers. Then

$$\forall \kappa\ [\kappa > w \to |f(\kappa)| < B\kappa^t]$$

is true for ${}^*\mathcal{R}$; so

$$\exists a \forall x\ [x > a \rightarrow |f(x)| < B\,x^t]$$

is true for ${}^*\mathcal{R}$, therefore this statement is true for $\mathcal{R}$ when it is interpreted in $\mathcal{R}$; but this means that $f = \mathrm{O}(x^t)$. □

Here is a useful corollary to Theorem 1.1.

1.2. COROLLARY. *Let $t \in R$, and let $f$ be an asymptotically finite function such that $\exists \kappa\ [|f(\kappa)| \geqslant \kappa^{t+1}]$, $\kappa$ positive and infinite. Then $f \neq \mathrm{O}(x^t)$.*

*Proof.* Assume for the moment that $f = \mathrm{O}(x^t)$. By Theorem 1.1, there is a standard number $B$ such that $\forall \kappa\ [|f(\kappa)| < B\,\kappa^t]$. For each positive infinite $\kappa$, $B\,\kappa^t < \kappa^{t+1}$; therefore $\forall \kappa\ [|f(\kappa)| < \kappa^{t+1}]$. This contradicts the hypothesis that $\exists \kappa\ [|f(\kappa)| \geqslant \kappa^{t+1}]$. We conclude that $f \neq \mathrm{O}(x^t)$. □

Here is a criterion for asymptotically finite functions.

1.3. THEOREM. *$f$ is asymptotically finite iff there are standard numbers $B$ and $t$ such that $\forall \kappa\ [|f(\kappa)| < B\,\kappa^t]$, $\kappa$ positive and infinite.*

*Proof.* (i) Assume that $f$ is asymptotically finite. Then $f = \mathrm{O}(x^t)$ for some $t \in R$; thus, by Theorem 1.1, there is a standard number $B$ such that $\forall \kappa\ [|f(\kappa)| < B\,\kappa^t]$.

(ii) Assume that there are standard numbers $B$ and $t$ such that $\forall \kappa\ [|f(\kappa)| < B\,\kappa^t]$. By Theorem 1.1, $f = \mathrm{O}(x^t)$. It remains to show that $\operatorname{dom} f$ includes a neighbourhood of $\infty$. By assumption, $\forall \kappa\ [\kappa \in \operatorname{dom} f]$; so

$$\forall \kappa\ [\kappa > w \rightarrow \kappa \in \operatorname{dom} f]$$

is true for ${}^*\mathcal{R}$, therefore

$$\exists a \forall x\ [x > a \rightarrow x \in \operatorname{dom} f]$$

is true for ${}^*\mathcal{R}$, so is true for $\mathcal{R}$ when interpreted in $\mathcal{R}$. We conclude that there is a standard number, say $a$, such that $\{t \in R \mid t > a\}$ is a subset of $\operatorname{dom} f$. □

We can refine this criterion a little more.

1.4. CRITERION FOR ASYMPTOTICALLY FINITE FUNCTIONS. *$f$ is asymptotically finite iff there is a standard number $t$ such that $\forall \kappa\ [|f(\kappa)| < \kappa^t]$, $\kappa$ positive and infinite.*

*Proof.* (i) Assume that $f$ is asymptotically finite. By Theorem 1.3, there are standard numbers $B$ and $t$ such that $\forall\kappa\ [|f(\kappa)| < B\kappa^t]$. For each positive infinite $\kappa$, $B\kappa^t < \kappa^{t+1}$ since $B > 0$. Thus $\forall\kappa\ [|f(\kappa)| < \kappa^{t+1}]$.

(ii) Assume $\forall\kappa\ [|f(\kappa)| < \kappa^t]$ for some $t \in R$. Then $f$ is asymptotically finite by Theorem 1.3 (with $B = 1$). □

Interesting function spaces can be obtained from $\mathcal{P}$ by imposing a third condition on the functions $f$ of the function space. For example, we can require one of the following:

(a) $f$ has an asymptotic expansion of the form $\Sigma a_i x^{-\nu_i}$, where $(\nu_i)$ is strictly increasing and $\lim(\nu_i) = \infty$;

(b) $f$ is continuous;

(c) $\operatorname{dom} f' = \operatorname{dom} f$.

(d) $\operatorname{dom} f^{(n)}$ is a neighbourhood of $\infty$ for each $n \in N$.

## 2. Convergence

There are several ways of characterizing convergence in a function space. Let $(x^{-\nu_i})$ be an asymptotic sequence of the sort considered in Section 6.3; i.e., $(\nu_i)$ is strictly increasing and $\lim(\nu_i) = \infty$. We want to introduce a notion of convergence under which $(\Sigma_0^n a_i x^{-\nu_i})$ converges to $f$ in case $f \sim \Sigma a_i x^{-\nu_i}$. Accordingly, we define our basic notion of convergence with one eye on the Nonstandard Criterion for Asymptotic Expansions 6.3.6.

2.1. DEFINITION OF CONVERGENCE. Let $(f_n)$ be a standard sequence of asymptotically finite functions, and let $f$ be a standard function. We say that $(f_n)$ *converges to* $f$, in symbols $\lim(f_n) = f$, provided that corresponding to each standard number $t$ there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,

$$\kappa^t\,[f(\kappa) - f_n(\kappa)] \simeq 0. \tag{2.2}$$

Here is an example.

2.3. EXAMPLE. We shall show that $\lim(x^{-n}) = 0$, the zero function. Let $t \in R$ and let $\kappa$ be positive and infinite; then

$$\kappa^t \kappa^{-n} = \kappa^{t-n} \simeq 0,$$

provided that $n > t$, $n \in N$. Therefore $\lim(x^{-n}) = 0$.

First, we shall prove that $f$ is asymptotically finite if $\lim(f_n) = f$.

2.4. LEMMA. *Let* $(f_n)$ *be a standard sequence of asymptotically finite functions, and let* $f$ *be a standard function such that* $\lim(f_n) = f$. *Then* $f$ *is asymptotically finite.*

*Proof.* By (2.2), with $t = 0$, there is a standard natural number, say $n$, such that

$$\forall\kappa\,[f(\kappa) - f_n(\kappa) \simeq 0],$$

so $\forall\kappa\,[f(\kappa) \simeq f_n(\kappa)]$. Here $f_n$ is asymptotically finite; by our Criterion 1.4, there is a standard number $t$ such that

$$\forall\kappa\,[|f_n(\kappa)| < \kappa^t].$$

Let $\kappa$ be any positive infinite number; then

$$f(\kappa) = f_n(\kappa) + \epsilon,$$

where $\epsilon \simeq 0$ and depends on $\kappa$. Therefore

$$\begin{aligned} |f(\kappa)| &\leqslant |f_n(\kappa)| + |\epsilon| && \text{by the Triangle Inequality} \\ &< \kappa^t + |\epsilon| \\ &< \kappa^{|t|+1}. \end{aligned}$$

This proves that

$$\forall\kappa\,[|f(\kappa)| < \kappa^{|t|+1}];$$

so, by Criterion 1.4, $f$ is asymptotically finite. □

We are really dealing with equivalence classes consisting of asymptotically equal functions. So we must show that the limit of a convergent sequence is unique up to asymptotic equality.

2.5. LEMMA. *Let* $(f_n)$ *be a standard sequence of asymptotically finite functions, and let* $f$ *and* $g$ *be standard functions such that* $\lim(f_n) = f$ *and* $\lim(f_n) = g$. *Then* $f \cong g$.

*Proof.* Choose $t \in R$. By (2.2) there are standard natural numbers $q_1$ and $q_2$ such that for each positive infinite $\kappa$,

$$\kappa^t\,[f(\kappa) - f_n(\kappa)] \simeq 0 \quad \text{if } n > q_1,$$

$$\kappa^t\,[g(\kappa) - f_n(\kappa)] \simeq 0 \quad \text{if } n > q_2.$$

Let $q = 1 + \max\{q_1, q_2\}$; then for each positive infinite $\kappa$,

$$\kappa^t\,[f(\kappa) - f_q(\kappa)] \simeq 0, \qquad \kappa^t\,[g(\kappa) - f_q(\kappa)] \simeq 0.$$

The difference of two infinitesimals is an infinitesimal; so

$$\kappa^t\,[f(\kappa) - g(\kappa)] \simeq 0,$$

and it follows that for each positive infinite $\kappa$,

$$|f(\kappa) - g(\kappa)| < \kappa^{-t} = \rho^t, \tag{2.6}$$

where $\rho = 1/\kappa$. But (2.6) is true for each $t \in R$; thus, for each $n \in N$, from (2.6),

$$|f(\kappa) - g(\kappa)| < \rho^n,$$

i.e., $f(\kappa) - g(\kappa)$ is an iota with respect to $1/\kappa$, for each positive infinite $\kappa$. So, by Criterion 6.3.18, $f \cong g$. This completes our proof. □

In this connection we mention that if $\lim(f_n) = f$ and $g \cong f$, then $\lim(f_n) = g$. To see this, choose $t \in R$. By (2.2), there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,

$$\kappa^t\,[f(\kappa) - f_n(\kappa)] \simeq 0.$$

Also, for each positive infinite $\kappa$, $f(\kappa) \approx g(\kappa)$ with respect to $1/\kappa$. So

$$\kappa^t\,|g(\kappa) - f(\kappa)| < \kappa^t \rho^m = \rho^{m-t} \quad (\rho = 1/\kappa),$$

where $m > t$, $m \in N$. Thus, for each positive infinite $\kappa$,

$$\kappa^t\,[g(\kappa) - f(\kappa)] \simeq 0.$$

Therefore, for $n > q$, $n \in N$,

$$\kappa^t\,[g(\kappa) - f_n(\kappa)] = \kappa^t\,[g(\kappa) - f(\kappa)] + \kappa^t\,[f(\kappa) - f_n(\kappa)] \simeq 0$$

since the sum of two infinitesimals is an infinitesimal. We conclude, by (2.2), that $\lim(f_n) = g$.

Similarly, we must show that two sequences converge to the same function if corresponding terms of the sequences are asymptotically equal.

2.7. LEMMA. *Let* $(f_n)$ *and* $(g_n)$ *be standard sequences of asymptotically finite functions such that* $\lim(f_n) = f$ *and* $f_i \cong g_i$ *for each* $i \in N$. *Then* $\lim(g_n) = f$.

*Proof.* We are given that $f_i(\kappa) \approx g_i(\kappa)$, with respect to $1/\kappa$, for each positive infinite $\kappa$ and for each $i \in N$. Fix $t \in R$; by assumption, there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,

$$\kappa^t \, [f(\kappa) - f_n(\kappa)] \simeq 0. \tag{2.8}$$

Also, from the comment that begins this proof,

$$|f_n(\kappa) - g_n(\kappa)| < \kappa^{-m}, \tag{2.9}$$

where $m > t$, $m \in N$. We are now ready to show that $\lim(g_n) = f$. For each $n > q$, $n \in N$, and for each positive infinite $\kappa$,

$$\begin{aligned} \kappa^t \, [f(\kappa) - g_n(\kappa)] &= \kappa^t \, [f(\kappa) - f_n(\kappa)] + \kappa^t \, [f_n(\kappa) - g_n(\kappa)] \\ &\simeq \kappa^t \, [f_n(\kappa) - g_n(\kappa)] \end{aligned}$$

by (2.8). But

$$\kappa^t \, |f_n(\kappa) - g_n(\kappa)| < \kappa^{t-m}$$

by (2.9), and $\kappa^{t-m}$ is an infinitesimal, so $\kappa^t \, [f_n(\kappa) - g_n(\kappa)] \simeq 0$. Thus, for each $n > q$, $n \in N$, and for each positive infinite $\kappa$,

$$\kappa^t \, [f(\kappa) - g_n(\kappa)] \simeq 0.$$

We conclude that $\lim(g_n) = f$.

The following fact follows from the Definition of Convergence 2.1.

2.10. LEMMA. $\lim(f_n) = f$ *iff* $\lim(f_n - f) = 0$, *the zero function.*

We now consider the comment at the beginning of this section which motivated our definition of convergence. In the following theorem, $(\nu_i)$ is a strictly increasing sequence of standard numbers such that $\lim(\nu_i) = \infty$.

2.11. THEOREM. $f \sim \Sigma\, a_i x^{-\nu_i}$ *iff* $\lim(\Sigma_0^n\, a_i x^{-\nu_i}) = f$.

*Proof.* Let $f_n = \Sigma_0^n\, a_i x^{-\nu_i}$ for each $n \in N$.

(i) Assume that $f \sim \Sigma\, a_i x^{-\nu_i}$. By our Nonstandard Criterion for Asymptotic Expansions 6.3.6,

$$\forall t\, \exists q\, \forall n \kappa\, [n > q \rightarrow \kappa^t \, [f(\kappa) - f_n(\kappa)] \simeq 0] \qquad (t \in R, q, n \in N, \ \kappa \text{ positive and infinite}),$$

so $\lim(f_n) = f$, i.e., $\lim(\Sigma_0^n\, a_i x^{-\nu_i}) = f$.

(ii) Assume that $\lim(f_n) = f$. By our Definition of Convergence 2.1,

$$\forall t\, \exists q\, \forall n\kappa \left[n > q \to \kappa^t \left[f(\kappa) - \sum_0^n a_i \kappa^{-\nu_i}\right] \simeq 0\right] \qquad (t \in R,\, q,n \in N,\ \kappa \text{ positive and infinite}),$$

so, by (6.3.7), $f \sim \Sigma a_i x^{-\nu_i}$. This completes our proof. □

In Section 3 we shall present a method of characterizing convergence which involves the valuations $v_\kappa$ on ${}^\rho\mathcal{R}$ introduced in Section 6.3, one valuation for each positive infinite $\kappa$. To prepare the way, we present the following fact.

2.12. LEMMA. *Let* $\lim(f_n) = 0$, *where each* $f_n$ *is asymptotically finite. Then*

$$\forall t\, \exists q\, \forall n\kappa\, [n > q \to |f_n(\kappa)|_{v_\kappa} \leqslant e^{-t}] \qquad (t \in R,\, q,n \in N,\ \kappa \textit{ positive and infinite}).$$

*Proof*. By (2.2),

$$\forall t\, \exists q\, \forall n\kappa\, [n > q \to \kappa^t f_n(\kappa) \simeq 0] \qquad (t \in R,\, q,n \in N\ \ \kappa \text{ positive and infinite}).$$

Thus, for $n > q$, $|f_n(\kappa)| < \kappa^{-t}$ uniformly for each positive, infinite $\kappa$. Therefore

$$\log_\rho |f_n(\kappa)| > t, \quad \rho = 1/\kappa.$$

So

$$v_\kappa(f_n(\kappa)) = {}^0(\log_\rho |f_n(\kappa)|) \geqslant t,$$

thus

$$-v_\kappa(f_n(\kappa)) \leqslant -t,$$

hence

$$\exp\,[-v_\kappa(f_n(\kappa))] \leqslant e^{-t},$$

i.e.,

$$|f_n(\kappa)|_{v_\kappa} \leqslant e^{-t}.$$

This establishes our lemma. □

## 3. Norm

In Section 2, we characterized convergence by means of an epsilon–delta type statement; this provided us with a direct characterization of the concept. Just as in the case of a field (see Section 1.5), convergence can be expressed in terms of a metric $d$ on our function space, as follows. Let $(f_n)$ be a standard sequence of asymptotically finite functions, and let $f$ be a standard function; then we say that $\lim(f_n) = f$ provided that $\lim(d(f_n, f)) = 0$. In this way, the question of the convergence of $(f_n)$ is reduced to the question of the convergence of a standard sequence of standard numbers, the sequence $(d(f_n, f))$.

It is well known that a metric $d$ is yielded by a norm $\|f\|$ on the function space, where $d(f, g)$ is defined to be $\|f-g\|$. We want to introduce a norm on our function space that yields, via the associated metric $d$, the same notion of convergence as our direct characterization (2.2).

3.1. DEFINITION OF NORM. For each $f \in P$,

$$\|f\| = \sup_{\kappa} |f(\kappa)|_{v_\kappa},$$

where $\kappa$ is positive and infinite.

A *norm* on a function space is a map into $R$ with the following properties:

(a) $\|f\| = 0$ if $f \cong 0$, the function 0;
(b) $\|f\| > 0$ for each $f \not\cong 0$;
(c) $\|-f\| = \|f\|$ for each $f$;
(d) $\|f+g\| \leqslant \|f\| + \|g\|$ for each $f$ and $g$ (Triangle Inequality).

We can establish these facts about our map $\| \ \|$ by applying the properties of $| \ |_\kappa$ listed in Section 1.5. We do this in Section 4.

Our definition of norm involves the valuations $v_\kappa$ (see Section 6.3) one for each positive infinite $\kappa$. With each valuation $v_\kappa$ we associate $| \ |_{v_\kappa}$, a map of ${}^\rho R$ into $R$ (see Section 1.5). Remember, it is only as a matter of convenience that we apply a valuation $v_\kappa$ to a member of ${}^*R$; $v_\kappa$ is a valuation on ${}^\rho R$ ($\rho = 1/\kappa$). Moreover, as defined in Section 1.5, $|a|_{v_\kappa} = e^{-v_\kappa(a)}$ for each $a \in {}^\rho R$; here, $e^{-\infty}$ is identified with 0. We shall abbreviate $| \ |_{v_\kappa}$ by writing $| \ |_\kappa$.

Since we are really dealing with equivalence classes of functions, it is important to prove that asymptotically equal functions have the same norm. First we point out that if $a \approx b$ with respect to $1/\kappa$, then $v_\kappa(a) = v_\kappa(b)$. Let $f \cong g$; so, for each positive infinite $\kappa$, $f(\kappa) \approx g(\kappa)$ with respect to $1/\kappa$. Therefore $v_\kappa(f(\kappa)) = v_\kappa(g(\kappa))$ for each positive infinite $\kappa$; hence

$$|f(\kappa)|_\kappa = |g(\kappa)|_\kappa$$

for each positive infinite $\kappa$. Thus

$$\|f\| = \sup_\kappa |f(\kappa)|_\kappa = \sup_\kappa |g(\kappa)|_\kappa = \|g\|.$$

Before going on, let us take a moment to illustrate this notion of convergence.

3.2. EXAMPLE. Let us show that $\lim(x^{-n}) = 0$ under the approach to convergence based on the above norm. For each $n \in N$,

$$\|x^{-n}\| = \sup_{\kappa} |\kappa^{-n}|_{\kappa} = \sup_{\kappa} \exp[-v_{\kappa}(\kappa^{-n})] = \sup_{\kappa} e^{-n} = e^{-n}.$$

Therefore,

$$\lim(d(x^{-n}, 0)) = \lim(\|x^{-n}\|) = \lim(e^{-n}) = 0.$$

So, under this criterion for convergence, $\lim(x^{-n}) = 0$, the zero function.

How do our two notions of convergence compare? It turns out that they yield the same concept of convergence in our function space. Indeed, we shall prove that for each sequence of asymptotically finite functions $(f_n)$ and for each standard function $f$, $\lim(f_n) = f$ under the criterion for convergence given by (2.2) iff $\lim(f_n) = f$ under the criterion for convergence provided by the metric $d$ yielded by our norm $\| \ \|$.

Let us show, first, that Lemma 2.10 is correct for the notion of convergence based on our metric $d$.

3.3. LEMMA. $\lim(f_n) = f$ *iff* $\lim(f_n - f) = 0$, *the zero function.*

*Proof.* In terms of our metric $d$,

$$\begin{aligned} \lim(f_n) = f &\leftrightarrow \lim(d(f_n, f)) = 0 \\ &\leftrightarrow \lim(\|f_n - f\|) = 0 \\ &\leftrightarrow \lim(d(f_n - f, 0)) = 0 \\ &\leftrightarrow \lim(f_n - f) = 0. \end{aligned}$$

This completes our proof. □

So, under either notion of convergence, $\lim(f_n) = f$ iff $\lim(f_n - f) = 0$. Therefore we can prove that both approaches yield the same concept of convergence by showing that, for each sequence $(f_n)$ of asymptotically finite functions, $\lim(f_n) = 0$ under both approaches, or under neither approach.

Under the metric approach,

$$\begin{aligned} \lim(f_n) = 0 &\leftrightarrow \lim(d(f_n, 0)) = 0 \\ &\leftrightarrow \lim(\|f_n\|) = 0. \end{aligned}$$

We shall use this fact in our proof.

3.4. THEOREM. *Let* $(f_n)$ *be any sequence of asymptotically finite functions. Then* $\lim(f_n) = 0$ *under* (2.2) *iff* $\lim(f_n) = 0$ *with respect to the metric d.*

*Proof.* (i) Assume that $\lim(f_n) = 0$ under (2.2). We shall prove that $\lim(\|f_n\|) = 0$ (in $\mathcal{R}$). Choose $t \in R$; by Lemma 2.12, there is a standard natural number $q$ such that

$$\forall n\kappa\, [n > q \to |f_n(\kappa)|_\kappa \leqslant e^{-t}] \qquad (n \in N,\ \kappa \text{ positive and infinite})$$

thus

$$\forall n\, [n > q \to \sup_\kappa |f_n(\kappa)|_\kappa \leqslant e^{-t}] \quad (n \in N).$$

Therefore, by the Definition of Norm 3.1,

$$\forall n\, [n > q \to \|f_n\| \leqslant e^{-t}] \qquad (n \in N).$$

We conclude that $\lim(\|f_n\|) = 0$.

(ii) Assume that $\lim(\|f_n\|) = 0$; i.e.,

$$\lim_\infty(\sup_\kappa |f_n(\kappa)|_\kappa) = 0.$$

Corresponding to each standard number $t$, there is a standard natural number $q$ such that for each $n > q$, $n \in N$, and $\kappa$ positive and infinite,

$$\sup_\kappa |f_n(\kappa)|_\kappa < e^{-t-1}.$$

Thus, for each positive infinite $\kappa$,

$$|f_n(\kappa)|_\kappa < e^{-t-1}.$$

But

$$|f_n(\kappa)|_\kappa = \exp[-v_\kappa(f_n(\kappa))] = \exp[-{}^0(\log_\rho |f_n(\kappa)|)] < e^{-t-1}, \quad \rho = 1/\kappa.$$

Thus

$$-{}^0(\log_\rho |f_n(\kappa)|) < -t - 1, \qquad {}^0(\log_\rho |f_n(\kappa)|) > t + 1,$$

so

$$\log_\rho |f_n(\kappa)| > t + 1,$$

thus $|f_n(\kappa)| < \rho^{t+1}$, hence $\kappa^t\, |f_n(\kappa)| < \rho$, therefore

$$\kappa^t f_n(\kappa) \simeq 0.$$

This proves that

$$\forall t\, \exists q\, \forall n\kappa\, [n > q \to \kappa^t f_n(\kappa) \simeq 0] \qquad (t \in R, \quad q,n \in N,\ \kappa \text{ positive and infinite}).$$

By (2.2), we conclude that $\lim(f_n) = 0$. This completes our proof. □

This shows that for each sequence $(f_n)$ of asymptotically finite functions, and for each standard function $f$, $(f_n)$ converges to $f$ under the notion of convergence provided by (2.2), iff $(f_n)$ converges to $f$ with respect to the metric $d$, i.e., iff $\lim(\|f_n - f\|) = 0$. But $(\|f_n - f\|)$ is a sequence of standard numbers, indeed $(\|f_n - f\|)$ is a standard sequence; so we can characterize its convergence in nonstandard terms. This yields the following nonstandard criterion for convergence.

3.5. NONSTANDARD CRITERION FOR CONVERGENCE. *Let* $(f_n)$ *be a standard sequence of asymptotically finite functions, and let* $f$ *be a standard function. Then* $\lim(f_n) = f$ *iff* $\|f_\kappa - f\| \simeq 0$ *for each infinite natural number* $\kappa$.

We now wish to present another method of characterizing $\|f\|$, where $f$ is asymptotically finite. If $f \cong 0$, then $\|f\| = \|0\| = 0$. Accordingly, we concentrate on functions which are *not* asymptotically equal to 0. Recall that $f \cong 0$ iff $f(\kappa)$ is an iota (with respect to $1/\kappa$) for each positive infinite $\kappa$; so, if $f$ is not asymptotically equal to 0, there is a positive infinite $\kappa$ such that $f(\kappa)$ is not an iota (with respect to $1/\kappa$), and in particular $f(\kappa) \neq 0$. In this case, $\sup_\kappa {}^0(|f(\kappa)|^{1/(\ln \kappa)})$ exists, where we take the supremum over positive infinite $\kappa$ such that $f(\kappa) \neq 0$.

We shall prove that if $f$ is not asymptotically equal to 0, then

$$\|f\| = \overline{\lim_{t\to\infty}}\ |f(t)|^{1/(\ln t)}.$$

As a first step toward this result, we require the following fact.

3.6. LEMMA. $\|f\| = \sup_\kappa {}^0(|f(\kappa)|^{1/(\ln \kappa)})$ *if* $f$ *is not asymptotically equal to* 0.

*Proof.* By assumption, $f(\kappa) \neq 0$ for some positive infinite $\kappa$. By Lemma 6.3.19, if $f(\kappa) \neq 0$, then

$$|f(\kappa)|_\kappa = {}^0(|f(\kappa)|^{1/(\ln \kappa)}) \geqslant 0.$$

So

$$\|f\| = \sup_\kappa |f(\kappa)|_\kappa = \sup_\kappa {}^0(|f(\kappa)|^{1/(\ln \kappa)}).$$

3.7. NOTE. If $f(\kappa) = 0$, then $|f(\kappa)|_\kappa = |0|_\kappa = 0$.

Theorem 3.11 below depends on the fact that the limit superior at $\infty$ of a standard function can be characterized as the supremum of the standard parts of the values of the function, for values of its argument which are positive and infinite. We now prove this statement.

3.8. LEMMA. *For each standard function g,*

$$\overline{\lim_{\infty}}\, g = \sup_{\kappa}\, {}^0g(\kappa),$$

*where κ is positive and infinite.*

*Proof.* Let

$$L = \overline{\lim_{\infty}}\, g = \lim_{a\to\infty}\, \sup\{g(t) \mid t > a\}.$$

Then $L$ is characterized by the following two properties:

$$(3.9)\qquad \forall\epsilon\, \exists a\, \forall t\, [t > a \to g(t) < L + \epsilon] \quad (\epsilon, a, t \in R,\ \epsilon > 0),$$

$$(3.10)\qquad \forall\epsilon a\, \exists t\, [t > a \wedge g(t) < L - \epsilon] \quad (\epsilon, a, t \in R,\ \epsilon > 0).$$

We shall show that

$$\sup\{{}^0(g(\kappa)) \mid \kappa \text{ positive and infinite}\} = S$$

shares these properties. Now ${}^0(g(\kappa)) \leqslant S$ for each positive infinite $\kappa$. Thus, for each positive $\epsilon \in R$,

$$\exists q\, \forall t\, [t > q \to g(t) < S + \epsilon] \qquad (q, t \in {}^*R)$$

is true for ${}^*\mathcal{R}$, so is true for $\mathcal{R}$. Therefore

$$\forall\epsilon\, \exists q\, \forall t\, [t > q \to g(t) < S + \epsilon] \quad (\epsilon, q, t \in R,\ \epsilon > 0)$$

is true for $\mathcal{R}$; i.e., $S$ satisfies (3.9). Also, $S$ is the *least* upper bound of

$$\{{}^0(g(\kappa)) \mid \kappa \text{ positive and infinite}\};$$

so $S - \epsilon$ is *not* an upper bound of this set, for each positive $\epsilon \in R$. Therefore there is a positive infinite $\kappa$ such that ${}^0(g(\kappa)) > S - \epsilon$; so, for each positive $h \in R$, $h + g(\kappa) > S - \epsilon$. In particular, taking $\epsilon$ for $h$ yields $g(\kappa) > S - 2\epsilon$. Using $\frac{1}{2}\epsilon$ in place of $\epsilon$, i.e., choosing $\kappa$ that works for $\frac{1}{2}\epsilon$, we conclude that corresponding to each positive $\epsilon \in R$, there is a positive infinite $\kappa$ such that $g(\kappa) \geqslant S - \epsilon$; so

$$\forall q\, \exists t\, [t > q \wedge g(t) > S - \epsilon] \qquad (q, t \in {}^*R)$$

is true for ${}^*\mathcal{R}$, so is true for $\mathcal{R}$. Thus

$$\forall\epsilon q\, \exists t\, [t > q \wedge g(t) > S - \epsilon] \quad (\epsilon, q, t \in R,\ \epsilon > 0)$$

is true for $\mathcal{R}$; i.e., $S$ satisfies (3.10). Hence $L = S$. □

Here is our theorem.

3.11. THEOREM. *Let f be any asymptotically finite function which is not asymptotically equal to* 0. *Then*

$$\|f\| = \overline{\lim_{t\to\infty}}\ |f(t)|^{1/(\ln t)}.$$

*Proof.* If $f \not\cong 0$, then

$$\|f\| = \sup_{\kappa}\ {}^{0}(|f(\kappa)|^{1/(\ln \kappa)}) \qquad \text{by Lemma 3.6}$$

$$= \overline{\lim_{t\to\infty}}\ |f(t)|^{1/(\ln t)} \qquad \text{by Lemma 3.8}$$

since $|f|^{1/\ln}$ is a standard function.

## 4. Algebraic properties of the norm

We can establish many algebraic properties of $\|\ \|$ by appealing to corresponding properties of the maps $|\ |_\kappa$ (see Section 1.5). First we shall establish the four properties (a)–(d) of $\|\ \|$, listed in Section 3, which make this map a norm.

Let $f \cong 0$. Then $f(\kappa)$ is an iota with respect to $1/\kappa$ for each positive infinite $\kappa$. Thus $|f(\kappa)| < \rho^n$ for each $n \in N$, where $\rho = 1/\kappa$; so $v_\kappa(f(\kappa)) \geqslant n$ for each $n \in N$, thus $v_\kappa(f(\kappa)) = \infty$. So

$$|f(\kappa)|_\kappa = \exp[-v_\kappa(f(\kappa))] = e^{-\infty} = 0$$

for each positive infinite $\kappa$. Therefore

$$\|f\| = \sup_\kappa |f(\kappa)|_\kappa = 0.$$

If $f$ is not asymptotically equal to 0, there is a positive infinite $\kappa$ such that $f(\kappa)$ is not an iota with respect to $1/\kappa$. Thus $|f(\kappa)| > \kappa^{-n}$ for some $n \in N$, so

$$-v_\kappa(f(\kappa)) \geqslant -n,$$

hence

$$|f(\kappa)|_\kappa = \exp[-v_\kappa(f(\kappa))] \geqslant e^{-n},$$

therefore

$$\|f\| = \sup_\kappa |f(\kappa)|_\kappa \geqslant e^{-n} > 0.$$

This proves that $\|f\| > 0$ if $f \not\cong 0$.

By (2) of Section 1.5, $|-a|_\kappa = |a|_\kappa$ for each $a \in {}^*R$; so

$$\|-f\| = \sup_\kappa |-f(\kappa)|_\kappa = \sup_\kappa |f(\kappa)|_\kappa = \|f\|.$$

To verify the Triangle Inequality, observe that by (5) of Section 1.5,

$$|a+b|_\kappa \leqslant \max\{|a|_\kappa, |b|_\kappa\} \quad \text{for any } a,b \in {}^*R.$$

So

$$\|f+g\| = \sup_\kappa |f(\kappa)+g(\kappa)|_\kappa \leqslant \sup_\kappa \max\{|f(\kappa)|_\kappa, |g(\kappa)|_\kappa\}$$

$$\leqslant \max\{\|f\|, \|g\|\} \leqslant \|f\| + \|g\|.$$

Here are some more algebraic properties of $\| \ \|$.

4.1. LEMMA. *If* $\|f\| \leqslant \|g\|$, *then* $\|f+g\| \leqslant \|g\|$.

*Proof.*

$$\|f+g\| \leqslant \max\{\|f\|, \|g\|\} \qquad \text{as we have already seen}$$

$$\leqslant \|g\| \qquad \text{by assumption.} \ \square$$

4.2. LEMMA. $\|f\cdot g\| \leqslant \|f\|\cdot\|g\|$ *whenever f and g are asymptotically finite functions.*

*Proof.* By (4) of Section 1.5,

$$\forall ab\ [|ab|_\kappa = |a|_\kappa \cdot |b|_\kappa].$$

So

$$\|f\cdot g\| = \sup_\kappa |f(\kappa)\, g(\kappa)|_\kappa = \sup_\kappa (|f(\kappa)|_\kappa \cdot |g(\kappa)|_\kappa)$$

$$\leqslant (\sup_\kappa |f(\kappa)|_\kappa)\cdot(\sup_\kappa |g(\kappa)|_\kappa) = \|f\|\cdot\|g\|. \ \square$$

4.3. LEMMA. $\|cf\| = \|f\|$ *whenever f is asymptotically finite and* $c \neq 0$, $c \in R$.

*Proof.* Since $v_\kappa$ is a valuation on ${}^\rho\!R$,

$$v_\kappa(cf(\kappa)) = v_\kappa(c) + v_\kappa(f(\kappa))$$

$$= v_\kappa(f(\kappa))$$

since ${}^0(\log_p |c|) = 0$. Thus

$$|cf(\kappa)|_\kappa = \exp[-v(cf(\kappa))] = \exp[-v(f(\kappa))] = |f(\kappa)|_\kappa.$$

Therefore

$$\|cf\| = \sup_\kappa |cf(\kappa)|_\kappa = \sup_\kappa |f(\kappa)|_\kappa = \|f\|.$$

We mention that $\mathcal{P}$ is a *linear space*, indeed a *linear algebra*, since we can multiply our functions, so we can form the product of two equivalence classes. Although our norm yields a metric $d$ on this linear algebra, the resulting structure is *not* a normed linear algebra; this is due to the fact that our norm does not satisfy the *homogeneity* condition: $\|cf\| = |c| \cdot \|f\|$ for each asymptotically finite function $f$ and for each $c \in R$ (see Lemma 4.3). □

## 5. Popken's description of the norm

In 1953, J. Popken investigated $\mathcal{P}$, the space of asymptotically finite functions. Popken used a norm $\phi$ defined as follows:

$$\phi(f) = \mathrm{e}^\lambda$$

where $\lambda = \inf\{t \in R \mid f = O(x^t)\}$ and $f$ is any asymptotically finite function.

5.1. EXAMPLE. We shall compute $\phi(x)$. Observe that $x = O(x^t)$ iff $t \geqslant 1$. Thus

$$\inf\{t \in R \mid x = O(x^t)\} = 1;$$

i.e., $\lambda = 1$. So

$$\phi(x) = \mathrm{e}^1 = \mathrm{e},$$

Notice that

$$\|x\| = \sup_\kappa |\kappa|_\kappa = \sup_\kappa \exp[-v_\kappa(\kappa)] = \sup_\kappa \mathrm{e} = \mathrm{e}$$

since $v_\kappa(\kappa) = -1$ for each positive infinite $\kappa$. So $\phi(x) = \|x\|$.

Let us establish some properties of $\phi$.

5.2. LEMMA. $\phi(f) = \phi(g)$ *iff* $f \cong g$.

*Proof.* By assumption,

$$\forall \kappa\, [f(\kappa) \approx g(\kappa)];$$

i.e.,

$$\forall \kappa n\, [|f(\kappa) - g(\kappa)| < \kappa^{-n}] \quad (n \in N).$$

Let $f = O(x^t)$, where $t \in R$. By Theorem 1.1, there is a standard number $B$ such that $\forall \kappa\, [|f(\kappa)| < B\kappa^t]$. We shall show that $g = O(x^t)$. For each positive infinite $\kappa$,

$$\begin{aligned} |g(\kappa)| &\leqslant |g(\kappa) - f(\kappa)| + |f(\kappa)| \\ &< \kappa^{-n} + B\kappa^t \\ &< (B+1)\kappa^t \end{aligned}$$

where $n \in N$. By Theorem 1.1, we conclude that $g = O(x^t)$. Since the relation $\cong$ is symmetric, this proves that

$$f = O(x^t) \quad \text{iff} \quad g = O(x^t),$$

for each $t \in R$. So

$$\{t \in R \mid f = O(x^t)\} = \{t \in R \mid g = O(x^t)\};$$

thus $\phi(f) = \phi(g)$. □

5.3. LEMMA. *If $f \cong 0$, then $\phi(f) = 0$.*

*Proof.* By Lemma 5.2, $\phi(f) = \phi(0)$. But $0 = O(x^t)$ for each $t \in R$. Thus

$$\inf\{t \in R \mid 0 = O(x^t)\} = -\infty;$$

so $\phi(0) = e^{-\infty} = 0$. Therefore $\phi(f) = 0$.

5.4. LEMMA. *If $f \not\cong 0$, then $\phi(f) > 0$.*

*Proof.* By assumption, $\exists \kappa\, [f(\kappa) \not\approx 0]$; i.e., there is a standard natural number $n$ such that

$$\exists \kappa\, [|f(\kappa)| > \kappa^{-n}].$$

So, by Corollary 1.2, $f \neq O(x^{-n-1})$. Thus

$$\lambda = \inf\{t \in R \mid f(t) = O(x^t)\} \geqslant -n - 1,$$

so

$$\phi(f) = e^{\lambda} \geqslant e^{-n-1} > 0. \ \square$$

The way is clear, now, to prove that the norm of this section is the same as the norm of Section 3; i.e., $\phi(f) = \|f\|$ for each asymptotically finite function $f$.

5.5. THEOREM. *Let $f$ be any asymptotically finite function. Then $\phi(f) = \|f\|$.*

*Proof.* If $f \cong 0$, then $\|f\| = 0$ and $\phi(f) = 0$. Accordingly, we can assume that $f \not\cong 0$. Let

$$\lambda = \inf\{t \in R \mid f = O(x^t)\};$$

so $\lambda \in R$.

(i) Let $t > \lambda$, $t \in R$. Then $f = O(x^t)$. By Theorem 1.1,

$$\forall \kappa\, [|f(\kappa)| < B\kappa^t],$$

where $B$ is standard and depends on $t$. It follows that $v_\kappa(f(\kappa)) \geqslant -t$, so $-v_\kappa(f(\kappa)) \leqslant t$, thus

$$|f(\kappa)|_\kappa = \exp[-v_\kappa(f(\kappa))] \leqslant e^t.$$

In words, $e^t$ is an upper bound of

$$\{|f(\kappa)|_\kappa \mid \kappa \text{ positive and infinite}\}.$$

So

$$\sup_\kappa |f(\kappa)|_\kappa \leqslant e^t;$$

i.e., $\|f\| \leqslant e^t$.

(ii) Let $s < \lambda$, $s \in R$. Then $f \neq O(x^s)$; so, by Theorem 1.1,

$$\neg \exists B\, \forall \kappa\, [|f(\kappa)| < B\kappa^s],$$

i.e.,

$$\forall B\, \exists \kappa\, [|f(\kappa)| \geqslant B\kappa^s];$$

in particular, $|f(\kappa)| \geqslant \kappa^s$ for some positive infinite $\kappa$. It follows that $-v_\kappa(f(\kappa)) \geqslant s$, so $|f(\kappa)|_\kappa \geqslant e^s$; therefore $\|f\| \geqslant e^s$.

Summarizing (i) and (ii), if $s < \lambda < t$, then

$$e^s \leqslant \|f\| \leqslant e^t.$$

By the Intermediate Value Theorem, there is a standard number, say $u$, between $s$ and $t$ such that $\|f\| = e^u$. We claim that $u = \lambda$. To see this, suppose that $u < \lambda$. Take $s_0 = \frac{1}{2}(u + \lambda)$; then $u < s_0 < \lambda$. By (ii), $e^{s_0} \leqslant \|f\|$; so $s_0 \leqslant u$, but $u < s_0$. Finally suppose that $u > \lambda$. Take $t_0 = \frac{1}{2}(u + \lambda)$; so $\lambda < t_0 < u$. By (i) $\|f\| \leqslant e^{t_0}$; so $u \leqslant t_0$, but $t_0 < u$. This contradiction proves that $u = \lambda$. So $\|f\| = e^\lambda = \phi(f)$. This completes our proof. □

This result shows that our definition of $\| \ \|$, and the associated notion of convergence, are both canonical, i.e., independent of our particular choice of ${}^*\mathcal{R}$.

## 6. More properties of $\mathcal{P}$

We now present some properties of the function space $\mathcal{P}$ from the viewpoint of functional analysis. First we mention that if $f = \lim(f_n)$ and $g = \lim(g_n)$, then $f + g = \lim(f_n + g_n)$ and $f \cdot g = \lim(f_n \cdot g_n)$. The first statement is easy to prove; we shall prove the second statement. Now

$$\begin{aligned} \|fg - f_n g_n\| &= \|(f - f_n) \cdot g + f_n \cdot (g - g_n)\| \\ &\leqslant \|(f - f_n) \cdot g\| + \|f_n \cdot (g - g_n)\| \\ &\leqslant \|g\| \cdot \|f - f_n\| + \|f_n\| \cdot \|g - g_n\| \end{aligned}$$

by Lemma 4.2. Clearly, it is enough to show that the sequence $(\|f_n\|)$ is bounded; this follows, in the usual way, from the fact that $(f_n)$ converges. Indeed, $\lim(\|f_n - f\|) = 0$; so for $\epsilon > 0$,

$$\|f_n - f\| < \epsilon$$

if $n$ is sufficiently large, thus

$$\|f_n\| < \|f\| + \epsilon,$$

hence

$$\|f_n\| < 1 + \|f\|$$

if $n$ is sufficiently large. Therefore $(\|f_n\|)$ is bounded by $1 + \|f\|$; it follows that $f \cdot g = \lim(f_n \cdot g_n)$.

In terms of the Definition of Convergence 2.1, this result can be put as follows.

6.1. LEMMA. *Let* $\lim(f_n) = f$, *and let* $\lim(g_n) = g$. *Corresponding to each* $t \in R$ *there is a* $q \in N$ *such that for each* $n > q$, $n \in N$, *and for each positive infinite* $\kappa$:

$$\kappa^t \, [f(\kappa)\, g(\kappa) - f_n(\kappa)\, g_n(\kappa)] \simeq 0.$$

Next we shall show that each convergent sequence $(f_n)$ of asymptotically finite functions is a Cauchy sequence, i.e.,

$$(6.2) \qquad \forall h \, \exists q \, \forall mn \, [m,n > q \to \|f_m - f_n\| < h] \quad (h \in R,\ h > 0,\ q,m,n \in N),$$

or in nonstandard terms

(6.3) $\forall \kappa p\,[\|f_{\kappa+p} - f_\kappa\| \simeq 0] \quad (\kappa \in {}^*N - N,\ p \in {}^*N).$

Let $\lim(f_n) = f$. Then by the Nonstandard Criterion for Convergence 3.5, $\|f_\kappa - f\| \simeq 0$ for each $\kappa \in {}^*N - N$. Thus, by the Triangle Inequality,

$$\|f_{\kappa+p} - f_\kappa\| \leqslant \|f_{\kappa+p} - f\| + \|f_\kappa - f\| \simeq 0.$$

This establishes (6.3), so $(f_n)$ is a Cauchy sequence.

It is more difficult to prove the converse. A normed space in which each Cauchy sequence converges is said to be *complete* with respect to its norm.

6.4. THEOREM. *$\mathcal{P}$ is complete with respect to* $\|\ \ \|$.

*Proof.* We shall follow Popken's proof (see Popken [1953]). Let $(f_n)$ be any Cauchy sequence; so, corresponding to each $j \in N$ there is a standard natural number $q_j$ such that

$$\forall mn\,[m,n > q_j \to \|f_m - f_n\| < e^{-j-1}] \quad (m,n \in N).$$

Let $m,n > q_j$; then

$$\inf\{t \in R \mid f_m - f_n = O(x^t)\} < -j - 1,$$

so $f_m - f_n = O(x^{-j-1})$. It follows that to each $j \in N$ and $m,n > q_j$ there correspond a standard number $r_{mnj}$ such that

(6.5) $\forall x\,[x > r_{mnj} \to |f_m(x) - f_n(x)| < x^{-j}] \quad (x \in R).$

Moreover, we can choose each $q_{j+1} \geqslant \max\{j + 1, q_j\}$, so that the sequence $(q_j)$ is strictly increasing and unbounded.

The idea is to assign to $m$ the values $q_1, q_1 + 1, q_1 + 2, \ldots$ in turn. In (6.5), regard $m$ as fixed, $m \geqslant q_1$; let $n$ and $j$ vary subject to the conditions of (6.5) and the additional condition $n \leqslant m$. With $m$ fixed, there are only a finite number of choices for $n$ and $j$ satisfying these conditions, since the sequence $(q_j)$ is strictly increasing. Each of the resulting inequalities involves a value of $r_{mnj}$ ($m$ fixed). Define $r_m = \max_{n,j} r_{mnj}$. For each $j \in N$ such that $q_j < m$,

(6.6) $\forall x\,[x > r_m \wedge q_j < n \leqslant m \to |f_m(x) - f_n(x)| < x^{-j}] \quad (x \in R).$

Here $m$ takes the values $q_1, q_1 + 1, q_1 + 2, \ldots$ in turn; the values of $r_m$ are obtained from the values of $r_{mnj}$ associated with the inequalities of (6.5). Furthermore, we can assume that the sequence $(r_m)$ is strictly increasing and unbounded; this involves only a slight modification of our definition of each $r_m$.

We now turn to constructing the limit of $(f_n)$. Let

$$I_m = \{t \in R \mid r_m \leqslant t < r_{m+1}\}$$

for each $m$; then $\bigcup_m I_m$ is a neighbourhood of $\infty$. Consider the standard function $g = \bigcup_m f_m \mid I_m$; i.e.,

$$g(t) = f_m(t) \quad \text{if } r_m \leqslant t < r_{m+1}.$$

We shall prove that $g = \lim(f_n)$; i.e., $\lim(\|g - f_n\|) = 0$. Given $j \in N$, choose $q_j$ so that (6.6) is satisfied. Fix $n > q_j$ and choose $t \geqslant r_n$; then $t \in I_m$ for some $m \geqslant n$. Thus, by (6.6),

$$|g(t) - f_n(t)| = |f_m(t) - f_n(t)| < t^{-j}.$$

This proves that for each $n > q_j$ and for each $t \geqslant r_n$,

$$|g(t) - f_n(t)| < t^{-j}, \tag{6.7}$$

i.e., $g - f_n = O(x^{-j})$; so

$$\|g - f_n\| \leqslant e^{-j} \quad \text{if } n > q_j.$$

We conclude that $\lim(f_n) = g$. By Lemma 2.4, $g$ is asymptotically finite. This completes our proof of the theorem. □

In view of the remark which precedes Theorem 6.4, we obtain the following corollary.

6.8. LEMMA. *$(f_n)$ converges iff $(f_n)$ is a Cauchy sequence.*

We recall that $(f_n)$ is a Cauchy sequence iff $\|f_{\kappa+p} - f_\kappa\| \simeq 0$ for each infinite natural number $\kappa$ and each $p \in {}^*N$.

Our strong version of the Triangle Inequality, i.e.,

$$\|f + g\| \leqslant \max\{\|f\|, \|g\|\},$$

whenever $f$ and $g$ are asymptotically finite functions, allows us to establish a simple criterion for convergence of series. By mathematical induction, we obtain

$$\|f_1 + \ldots + f_n\| \leqslant \max_{1 \leqslant i \leqslant n} \|f_i\| \tag{6.9}$$

for each $n \in N$. Applying our Transfer Theorem 2.1.2 to (6.9) yields

$$\|f_1 + \ldots + f_n\| \leqslant \max_{1 \leqslant i \leqslant n} \|f_i\| \tag{6.10}$$

for each $n \in {}^*N$. □

We are now ready to prove the following fact, which is well known for any ultrametric (see Lemma 1.5.7).

6.11. CRITERION FOR CONVERGENCE OF SERIES. $\Sigma_N f_n$ *converges iff* $\lim(f_n) = 0$.

*Proof*. By definition,

$$\begin{aligned}
\sum_N f_n \text{ converges} \quad & \text{iff} \quad (f_1 + \ldots + f_n) \text{ converges} \\
& \text{iff} \quad (f_1 + \ldots + f_n) \text{ is a Cauchy sequence} \quad \text{(by Lemma 6.8)} \\
& \text{iff} \quad \|f_\kappa + \ldots + f_{\kappa+p}\| \simeq 0 \text{ for each } \kappa \in {}^*N - N,\ p \in {}^*N, \\
& \text{iff} \quad \lim(\|f_n\|) = 0 \qquad \text{(by (6.10) and the Nonstandard Criterion for Convergence 3.5)} \\
& \text{iff} \quad \lim(f_n) = 0. \ \square
\end{aligned}$$

This result has an immediate corollary.

6.12. COROLLARY. $(f_n)$ *converges iff* $\lim(f_{n+1} - f_n) = 0$.

*Proof*. Each sequence $(f_n)$ can be regarded as the sequence of partial sums of a certain series, namely the series $\Sigma_N g_n$, where $g_0 = f_0$ and $g_{n+1} = f_{n+1} - f_n$ for each $n \in N$. By our Criterion for Convergence 6.11 $\Sigma_N g_n$ converges iff $\lim(g_n) = 0$. Thus $(f_n)$ converges iff $\lim(f_{n+1} - f_n) = 0$. □

Naturally, we are specially interested in series of the form $\Sigma_N a_i x^{-\nu_i}$.

6.13. LEMMA. *Let* $a_n \neq 0$ *for each* $n \in N$. *Then* $\Sigma_N a_i x^{-\nu_i}$ *converges iff* $\lim(\nu_i) = \infty$.

*Proof*. By our Criterion for Convergence 6.11, $\Sigma_N a_i x^{-\nu_i}$ converges iff

$$0 = \lim(\|a_i x^{-\nu_i}\|) = \lim(\|x^{-\nu_i}\|) = \lim(e^{-\nu_i}).$$

So $\Sigma_N a_i x^{-\nu_i}$ converges iff $\lim(\nu_i) = \infty$; i.e., corresponding to each standard number $B$ there is a standard natural number $q$ such that

$$\forall n\, [n > q \rightarrow \nu_n > B] \quad (n \in N). \ \square$$

Turning to another idea, a series $\Sigma_N f_i$ is said to be *absolutely convergent* if $\Sigma_N \|f_i\|$ converges (in $\mathcal{R}$). For example, $\Sigma_N x^{-i}$ is absolutely convergent since $\|x^{-i}\| = e^{-i}$ for each $i \in N$, and $\Sigma_N e^{-i}$ converges. Applying our Criterion for Convergence, we see that $\Sigma_N x^{-i}$ converges. Let us prove that each absolutely convergent series converges; this is true in general (see Lemma 1.5.8).

6.14. LEMMA. *$\Sigma_N f_i$ converges if $\Sigma_N \|f_i\|$ converges.*

*Proof.* Given $\epsilon > 0$ and standard, for each $n \in N$ sufficiently large, and for each $p \geqslant 0$, $p \in N$,

$$\|f_n + \dots + f_{n+p}\| \leqslant \|f_n\| + \dots + \|f_{n+p}\| < \epsilon$$

ince $\Sigma_N \|f_i\|$ converges. Thus $\Sigma_N f_i$ converges. □

The converse is not true, i.e., there are convergent series which are not absolutely convergent. For example, $\Sigma_N x^{-\ln i}$ converges since $\lim(x^{-\ln i}) = 0$ (notice that $\|x^{-\ln i}\| = 1/i$). Clearly $\Sigma_N \|x^{-\ln i}\| = \Sigma_N 1/i$ which diverges. □

We mention that Lemmas 1.5.14 and 1.5.15 are true for any complete ring with a nonarchimedean valuation. Expressing these results for the function space $\mathcal{P}$, we obtain our next lemma.

6.15. LEMMA. *Let $\Sigma_N f_n = f$, and let $\Sigma_N g_n = g$. Then:*

(i) $\Sigma_N (f_n + g_n) = f + g$;

(ii) $\Sigma_N h_n = fg$, *where* $h_n = f_1 g_n + \dots + f_n g_1$ *for each* $n \in N$ *(this is the Cauchy product of the given series).*

As we have seen, each asymptotic expansion of the form $\Sigma a_i x^{-\nu_i}$, where $(\nu_i)$ is strictly increasing and $\lim(\nu_i) = \infty$, is also a convergent series. Accordingly, we can use Lemma 6.15 to establish the following facts about asymptotic expansions.

6.16. THEOREM. *Let $f \sim \Sigma a_i x^{-\nu_i}$, and let $g \sim \Sigma b_i x^{-\nu_i}$. Then:*

(i) $f + g \sim \Sigma (a_i + b_i) x^{-\nu_i}$;

(ii) $fg \sim \Sigma c_i x^{-2k-i}$ *provided that* $\nu_i = k + i$, *where* $k \in I$ *is fixed, and* $c_n = a_1 b_n + \dots + a_n b_1$ *for each* $n \in N$.

## 7. Asymptotic expansions in $\mathcal{P}$

We now present some basic facts about asymptotic expansions for asymptotically finite functions. Recall that if $a_n \in R$ and $a_n \neq 0$ for each $n \in N$, then $(a_i x^{-\nu_i})$ is an asymptotic sequence iff $(x^{-\nu_i})$ is an asymptotic sequence.

7.1. LEMMA. *Let $(x^{-\nu_i})$ be an asymptotic sequence, and let $a_n \neq 0$ for each $n \in N$. Then $\Sigma\, a_i x^{-\nu_i}$ is an asymptotic expansion for some function in P iff* $\lim(\nu_i) = \infty$.

*Proof.* By Theorem 2.11,

$$f \sim \sum a_i x^{-\nu_i} \quad \text{iff} \quad f = \sum_N a_i x^{-\nu_i}.$$

Moreover, by Lemma 6.13, $\Sigma_N\, a_i x^{-\nu_i}$ converges iff $\lim(\nu_i) = \infty$. By Lemma 2.4, if $\Sigma_N\, a_i x^{-\nu_i}$ converges then $\Sigma_N\, a_i x^{-\nu_i} \in P$. We conclude that $\Sigma\, a_i x^{-\nu_i}$ is an asymptotic expansion for some function iff $\lim(\nu_i) = \infty$.

If $\Sigma\, a_i x^{-\nu_i}$ is an asymptotic expansion for some function, then we can construct an asymptotically finite function $f \sim \Sigma\, a_i x^{-\nu_i}$ by applying the method given in the proof of Theorem 6.4, to the sequence of partial sums $(\Sigma_0^n\, a_i x^{-\nu_i})$. Here, once again, we utilize the fact that

$$f \sim \sum a_i x^{-\nu_i} \quad \text{iff} \quad f = \sum_N a_i x^{-\nu_i}.$$

This result can be generalized. Let $(\phi_i)$ be any asymptotic sequence (here the $\phi_i$ are not necessarily in $P$), and let $a_n \in R$, $a_n \neq 0$, for each $n \in N$. Then $(a_i \phi_i)$ is also an asymptotic sequence. We shall prove that the asymptotic series $\Sigma\, a_i \phi_i$ is an asymptotic expansion for some asymptotically finite function $f$. The following proof, which actually constructs a suitable $f$, is based on Erdélyi's adaptation of van der Corput's proof (see Erdélyi [1956]).

Since $a_{i+1}\phi_{i+1} = o(a_i \phi_i)$ for each $i \in N$, i.e.,

$$\lim_{\infty} \frac{|a_{i+1}\phi_{i+1}|}{|a_i \phi_i|} = 0,$$

we can construct a sequence of standard numbers $r_0, r_1, r_2, \ldots$ with the following properties:

(1) $\forall i t\, [t > r_i \to |a_{i+1}\phi_{i+1}(t)| < \frac{1}{2}|a_i \phi_i(t)|$ $\quad (i \in N,\ t \in R)$;

(2) $\forall i\, [r_{i+1} > 2r_i]$ $\quad (i \in N)$.

The second condition ensures that the sequence $(r_i)$ is strictly increasing and unbounded. As in our proof of completeness (see Section 6), let

$$I_m = \{t \in R \mid r_m \leqslant t < r_{m+1}\}$$

for each $m \in N$; so $\bigcup_m I_m$ is a neighbourhood of $\infty$.

Next define functions $\psi_0, \psi_1, \psi_2, \ldots$ such that:

(a) For each $m \in N$,

$$\psi_m(t) = \begin{cases} 0 & \text{if } t \leqslant r_m, \\ 1 & \text{if } t \geqslant r_{m+1}. \end{cases}$$

(b) For each $m \in N$, $\psi_m$ is increasing and continuous on $I_m$.

For each $m$, a suitable function $\psi_m$ exists and we may even choose it so that $\psi_m$ is infinitely differentiable. The point is that for each $t \in I_m$,

$$\psi_i(t) = \begin{cases} 0 & \text{if } i > m, \\ 1 & \text{if } i < m, \end{cases}$$

and $0 \leqslant \psi_m(t) \leqslant 1$. In view of this, the series $f = \Sigma_N a_n \psi_n \phi_n$ converges on $\bigcup_m I_m$. Indeed, let $t \in I_m$ for some $m$; then

$$(7.2) \qquad f(t) = \sum_N a_i \psi_i(t) \phi_i(t)$$
$$= a_0 \phi_0(t) + \ldots + a_{m-1} \phi_{m-1}(t) + a_m \psi_m(t) \phi_m(t).$$

So dom $f$ includes $\bigcup_m I_m$, a neighbourhood of $\infty$. We claim that $f \sim \Sigma a_n \phi_n$; by Criterion 6.1.3, it is enough to show that for each $n \in N$,

$$f - \sum_0^n a_i \phi_i = O(\phi_{n+1}).$$

Consider any $t > r_{n+1}$; there is a standard natural number $m$ such that

$m \geqslant n+1$ and $t \in I_m$. Thus

$$f(t)-\left|\sum_0^n a_i\phi_i(t)\right| = \left|a_0\phi_0(t)+\ldots+a_{m-1}\phi_{m-1}(t)+a_m\psi_m(t)\phi_m(t)-\sum_0^n a_i\phi_i(t)\right| \quad \text{by (7.2)}$$

$$= |a_{n+1}\phi_{n+1}(t)+\ldots+a_{m-1}\phi_{m-1}(t)+a_m\psi_m(t)\phi_m(t)|$$

$$\leqslant |a_{n+1}\phi_{n+1}(t)|+\ldots+|a_{m-1}\phi_{m-1}(t)|+|a_m\psi_m(t)\phi_m(t)|$$

$$\leqslant |a_{n+1}\phi_{n+1}(t)|+\ldots+|a_{m-1}\phi_{m-1}(t)|+|a_m\phi_m(t)| \quad \text{since } 0\leqslant \psi_m(t)\leqslant 1$$

$$< |a_{n+1}\phi_{n+1}(t)|(1+1/2+\ldots+1/2^{m-n}) \quad \text{by Property (1) of } (r_i)$$

$$< 2|a_{n+1}\phi_{n+1}(t)|.$$

So

$$f-\sum_0^n a_i\phi_i = \mathrm{O}(\phi_{n+1}).$$

This establishes our claim that $f \sim \Sigma\, a_n\phi_n$.

It remains to show that $f=\mathrm{O}(x^s)$ for some $s\in R$. Now $f-a_0\phi_0=\mathrm{O}(\phi_1)$; so, if $t>r_1$,

$$|f(t)-a_0\phi_0(t)|<2|a_1\phi_1(t)|.$$

Thus

$$|f(t)|<|a_0\phi_0(t)|+2|a_1\phi_1(t)|$$

$$< B\,t^a+C\,t^b \quad \text{by assumption, for some } a,b,B,C$$

$$<(B+C)\,t^s,$$

where $s=\max\{a,b\}$. So $f=\mathrm{O}(x^s)$. This completes our proof that $\Sigma_N\, a_i\,\psi_i\,\phi_i \sim \Sigma\, a_i\,\phi_i$. □

## 8. More about norms

Let $(\phi_i)$ be an asymptotic sequence, in $\mathcal{P}$, comparable to $(x^{-i})$. Let $f$ be a term of $(\phi_i)$ such that $f(t) > 0$ for each $t \in R$; moreover, assume that for each $s \in R$, $f < x^s$ or $x^s < f$, or $f \sim x^s$ in the sense of a Hardy field (see Section 1.6). Since $(\phi_i)$ is comparable to $(x^{-i})$, $f > x^{-i}$ for some $i \in N$. Moreover, $f$ is asymptotically finite, so $f = O(x^{-s-1})$ for some $s \in R$; thus $f < x^{-s}$. In short, $x^{-i} < f < x^{-s}$.

We want to compute $\|f\|$. If there is a standard number $t$ such that $f = x^t$, then $\|f\| = e^t$. Assume that $f \neq x^t$ for each $t \in R$. In this case, $f$ determines a Dedekind cut $(C_1, C_2) = r_0$ defined as follows:

$$r \in C_1 \quad \text{if } x^r < f,$$

$$r \in C_2 \quad \text{if } f < x^r.$$

Let $r_1 \in C_1$, and let $r_2 \in C_2$. Then for each positive infinite $\kappa$,

$$\kappa^{r_1} < f(\kappa) < \kappa^{r_2},$$

so

$$-r_2 \leqslant v_\kappa(f(\kappa)) \leqslant -r_1.$$

Since $C_1$ has no largest member and $C_2$ has no smallest member, it follows that

$$(8.1) \qquad -r_2 < v_\kappa(f(\kappa)) < -r_1.$$

We claim that $v_\kappa(f(\kappa)) = -r_0$. If $-v_\kappa(f(\kappa)) > r_0$, then $-v_\kappa(f(\kappa)) \in C_2$; so $v_\kappa(f(\kappa)) < v_\kappa(f(\kappa))$ by (8.1). Similarly, we obtain a contradiction by supposing that $-v_\kappa(f(\kappa)) \in C_1$. Thus, for each positive infinite $\kappa$,

$$|f(\kappa)|_\kappa = \exp[-v_\kappa(f(\kappa))] = e^{r_0}.$$

So

$$\|f\| = \sup_\kappa |f(\kappa)|_\kappa = e^{r_0}.$$

# BIBLIOGRAPHY


BACHMANN, George

[1964] *Introduction to p-adic Numbers and Valuation Theory* (Academic Press, New York).

BELL, J.L. and SLOMSON, A.B.

[1971] *Models and Ultraproducts: an Introduction* (North-Holland, Amsterdam).

BELLMAN, Richard

[1964] *Perturbation Techniques in Mathematics, Physics and Engineering* (Holt, Rinehart and Winston, New York).

BOURBAKI, N.

[1951] *Eléments de Mathématique*, Vol. 12, Book 4 (Hermann, Paris).

DE BRUIJN, N.G.

[1958] *Asymptotic Methods in Analysis* (North-Holland, Amsterdam).

CODDINGTON, E.A. and LEVINSON, N.

[1955] *Theory of Ordinary Differential Equations* (McGraw-Hill, New York).

COPSON, E.T.

[1935] *An Introduction to the Theory of Functions of a Complex Variable* (Oxford Univ. Press, London).

[1965] *Asymptotic Expansions* (Cambridge Univ. Press, London).

VAN DER CORPUT, J.G.

[1954] Asymptotic expansions. I. Fundamental theorems of asymptotics. Dept. of Mathematics, Univ. of California, Berkeley, Calif.

[1954a] Asymptotics. I, *Indag. Math.* **16**, 206–217.

[1955] Asymptotics. II. Elementary methods. *Indag. Math.* **17**, 139–150.

[1955/56] Asymptotic developments. I. Fundamental theorems of asymptotics. *J. Anal. Math.* 4, 341–418.

[1956] Asymptotics. IIIa. The asymptotic behaviour of the real solutions of certain second order differential equations. *Indag. Math.* **18**, 1–10.

[1956a] Asymptotics. IIIb. The asymptotic behaviour of the real solutions of certain second order differential equations. *Indag. Math.* **18**, 11–14.

DERRICK, W.R.

[1972] *Introductory Complex Analysis and Applications* (Academic Press, New York).

ERDÉLYI, A.

[1956] *Asymptotic Expansions* (Dover, New York).

[1961] Asymptotic solutions of ordinary linear differential equations. Mimeographed Notes, California Inst. of Technol., Pasadena, Calif.

FRIEDRICHS, K.O.

[1955] Asymptotic phenomena in mathematical physics. *Bull. Am. Math. Soc.* **61**, 485–504.

HARDY, G.H.

[1954] *Orders of Infinity: The 'Infinitärcalcül' of Paul du Bois-Reymond* (Cambridge Univ. Press, London).

JEFFREYS, Harold

[1962] *Asymptotic Approximations* (Oxford Univ. Press, London).

JEFFREYS, Harold and JEFFREYS, B.S.

[1956] *Methods of Mathematical Physics* (Cambridge Univ. Press, London).

LANGER, R.E.

[1934] The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to the Stokes phenomenon. *Bull. Am. Math. Soc.* 40, 545–582.

LAUGWITZ, D.

[1968] Eine nichtarchimedische Erweiterung angeordneter Körper, *Math. Nachr.* 37, 225–236.

LEVI-CIVITA, T.

[1892/93] Sugli infiniti ed infinitesimali actuali quali elementi analytici. *Opere Matematiche*, Vol. 1 (Bologna, 1954) 1–39.

LIGHTSTONE, A.H.

[1966] *Concepts of Calculus*, II (Harper and Row, New York).

[1972] Infinitesimals. *Am. Math. Monthly* 79, 242–251.

LUXEMBURG, W.A.J.

[1962] Non-standard analysis. Lectures on A. Robinson's theory of infinitesimals and infinitely large numbers. Mimeographed Notes. California Inst. of Technol., Pasadena, Calif.

OSTROWSKI, A.

[1935] Untersuchungen zur arithmetischen Theorie der Körper. *Math. Z.* 39, 269–404.

POINCARE, H.

[1886] Sur les intégrales irrégulières des équations linéaires. *Acta Math.* 8, 295–344.

POPKEN, J.

[1953] Asymptotic expansions from an algebraic standpoint. *Indag. Math.* 15, 131–143.

ROBINSON, Abraham

[1961] Non-standard analysis. *Proc. Koninkl. Nederl. Akad. Wetensch.* (A) 64, 432–440.

[1963] *Introduction to Model Theory and to the Metamathematics of Algebra* (North-Holland, Amsterdam).

[1966] *Non-standard Analysis* (North-Holland, Amsterdam).

[1967] Non-standard theory of Dedekind rings. *Indag. Math.* 29, 444–452.

[1970] Elementary embeddings of fields of power series. *J. Number Theory* 2, 237–247.

[1972] On the real closure of a Hardy Field. In: G. Asser et al., eds., *Theory of Sets and Topology*, Hausdorff Memorial Volume (VEB Deutscher Verlag der Wissensch., Berlin) 427–433.

[1973] Function theory on some nonarchimedean fields, *Am. Math. Monthly, Papers in the Foundations of Mathematics*, 80, 87–109.

WASOW, Wolfgang

[1965] *Asymptotic Expansions for Ordinary Differential Equations* (Wiley, New York).

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